Applications (Ideas) in Linear Algebra with Digital Image Processing. Can we Do, Teach, Motivate and Evaluate?

C. M. R. Caridade [1]*

[1] *Department of Mathematics and Physics, Polytechnic of Coimbra-Institute of Engineering*, Coimbra, PORTUGAL

***Corresponding Author:** caridade@isec.pt

ABSTRACT

The application of new methodologies in higher education brings with it questions such as: "Can we do it?", "Can we teach?", "Can we motivate students?", "Can we evaluate?". During the last 8 years, in the Linear Algebra course of the Mechanical and Electromechanical Engineering of the Polytechnic of Coimbra-Institute of Engineering, it has been trying to answer these questions. The teaching methodology presented consists in the approach of Linear Algebra contents, using digital Image Processing applications. This new environment is intended to encourage students to learn the concepts and their manipulation in a productive and dynamic way. From the analysis of the studies and the student's opinions (students' adherence and motivation, questionnaires), it has been found that the use of this new teaching methodology helps the 'learning to learn' as long as it is well defined, with clear objectives for those who implements (teacher) and who interacts (students) with it.

Keywords: teaching mathematics, linear algebra, image processing

INTRODUCTION

As in many other universities, Polytechnic of Coimbra-Institute of Engineering (PC-IE) offers basic curricular units of mathematics in the first year of all engineering. This curricular units are responsible for theoretical basis necessary for professional's future in Engineering, and for this they are present in most of the degrees taught in various higher education institutions. For students the basic math contents are difficult especially if they are abstract and theoretical. Fortunately, much of the mathematical content can be presented using practical applications to justify their study and arouse interest from students. The manipulation of mathematical concepts and their connection can be explored using methodological strategies in mathematical disciplines. An appropriate teaching strategy that allows the student to develop his or her critical spirit, understand the main concepts and apply them to a variety of situations is very important for effective learning (Caridade, 2019).

In view of the above findings, during the last 8 years, in Linear Algebra course (LA), mathematical content has been explored through practical applications of digital image processing. The Digital Image Processing (DIP) is a set of techniques and methods applied to digital images very useful nowadays. The algorithms developed in digital images are extremely rich in mathematical concepts and are one interesting pedagogical tool to explore LA concepts (Gonzalez et al., 2009). Some experiences have been made, for example the work presented by Professor Silverman and Rosen (2010) that uses the DIP and LA to teach concepts and terminologies of LA while developing self-confidence students. The DIP helps to develop visual and intuitive understanding of the concepts, which are usually abstract and entirely new for students (Berriochoa et al., 2009; Caridade et al., 2014; Donevska-Todorova,

Table 1. Correspondence between LA and DIP

Linear Algebra	Image Processing
Basic concepts of matrices	Image representation
Matrix operations and proprieties	Image operations and proprieties
Linear combination	Image operations
Geometric transform	Polygons matrix operations
Block matrix manipulation and operations	Block image operations
Random matrices	Create random images; create videos
Eigenvectors and eigenvalues	Image compression, Image segmentation

2018; Garía and Cano, 2018). So, the DIP can improve the teaching of LA in a completely different way by increasing the motivation and participation.

LA concepts like linear equations, algebra matrix, matrix operations, matrix transformations, and inverse transformation can be explored using applications of DIP (image representation, image transformation, image enhancement, image restoration, image segmentation or image compression) (Zin et al., 2015). Image filtering techniques are usually applied to increase the quality of magnetic resonance images, segmentation of skin cancer in dermoscopy images, edge detection techniques for fingerprint images, etc. All this Image Processing techniques involve LA subjects. The concepts to be learned by the students must be meaningful for them, so that they can assimilate it. Using matrices operations to eliminate the image background or identify parts of the image can enrich teaching practices and improve student learning. The use of the 2D-geometric transformation can bring one image into alignment with another increasing the motivation and improvement of student's skills in these concepts. LA and DIP can be taught simultaneously. With these combined lessons, it is hoped that students can overcome the difficulties in learning LA through the real-world applications. Furthermore, teachers are also able to teach the contents of LA to students in a more stimulating and motivating environment.

The work presented in this paper is a collection of applications developed for engineering students where LA contents are learned through DIP. Students' opinions on these initiatives are also presented in order to evaluate their pedagogical and motivational interest as well as identify possible changes to make to Curricular Unit (UC). The paper is organized in 6 parts. A first part with the introduction, in the second part is presented the Linear Algebra Curricular Unit (LA-UC) and in the third the methodology applied to LA-UC. In the fourth part monitoring and evaluation, in the fifth part the results and finally in the last part are presented the conclusions and future work.

LINEAR ALGEBRA CURRICULAR UNIT

The LA-CU is available to all students of the first year in Mechanical and Electromechanical Engineering degree in Polytechnic of Coimbra-Institute of Engineering. During the theoretical and practical lessons, a new teaching methodology are presented to the students, using a stimulating and interesting learning environment where DIP interacts with LA. To enrich this interconnection a practical project developed in Matlab is presented to the students. The contents of LA are presented to students as practical applications developed with DIP techniques. In **Table 1** it is possible to observe the correspondence of some of the LA contents with DIP.

CURRICULAR UNIT METHODOLOGY

A digital image is a two-dimensional discrete function $f(x, y)$ with $(x, y)\epsilon Z^2$ where each pair of coordinates is called a pixel. Without loss of generality, one image can be represented by a matrix, named for example A where each element a_{ij} corresponds to the value of the pixel image position (i, j) (Gonzalez et al., 2009). The binary image is a matrix where $f(x, y)$ is zero (black color) or one (white color). The grayscale image is also a matrix with integer values that measures the pixel intensity, so $f(x, y)$ usually set from 0 (black color) to 255 (white color). The color image (in this case RGB space) are a compose of three matrices that represent Red, Green and Blue space color respectively. The $f(x, y)$ value is a vetor with three parameters that correspond to the three spaces with integer values from 0 to 255 (Gonzalez et al., 2009). In **Figure 1** images are represented by binary, grayscale and color form. The students can calculate the size of the images, the values of each pixel in the different forms of the image´s representation and can convert the color image to grayscale and then to binary images. All the examples presented in this paper were made in the practical project by the students.

Figure 1. Two digital images repesentations: rgb image (left); grayscale image (center); binary image (right) (Thetapestryhouse.com, n.d.; wikipedia.org, n.d.a)
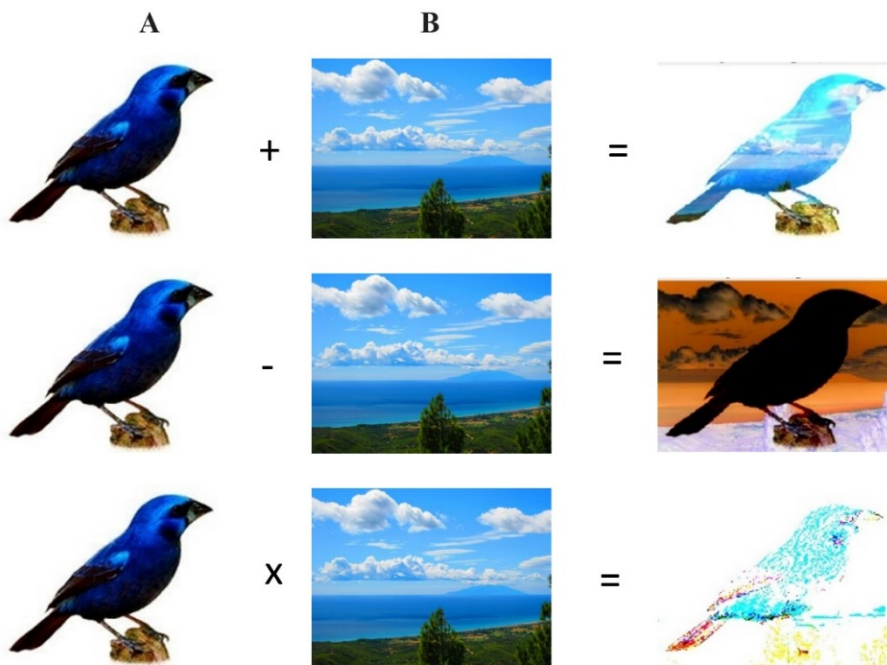

Figure 2. Some matrix operations using digital images (Produto.mercadolivre.com.br, n.d.)

All standard forms to manipulate images can be obtained by applying mathematical operations to the matrix associated with images (Bovik, 2000). Students build new images (matrices) by using elementary operations between matrices. Test their properties and constraints. Thus, it is possible with the images, for example, to verify the associative property between 3 matrices and the fact that the sum is only possible between matrices of the same dimensions. In **Figure 2** and **3** some operations between images are presented.

Linear combinations and linear systems are central concepts in LA. Therefore, the experience in the realization of linear combinations between matrices is of extreme importance for the learning of this concept. In this way students are asked to develop linear combinations between two different matrices but with the same dimensions. In **Figure 4**, a linear combination of two images were proposed. The image, A (the first one) and B (the last one), are combined using the equation $(1-k)A + kB$, white $k = 1/20$. This is a beautiful example where the matrices operations can be applied, and the result is amazing.
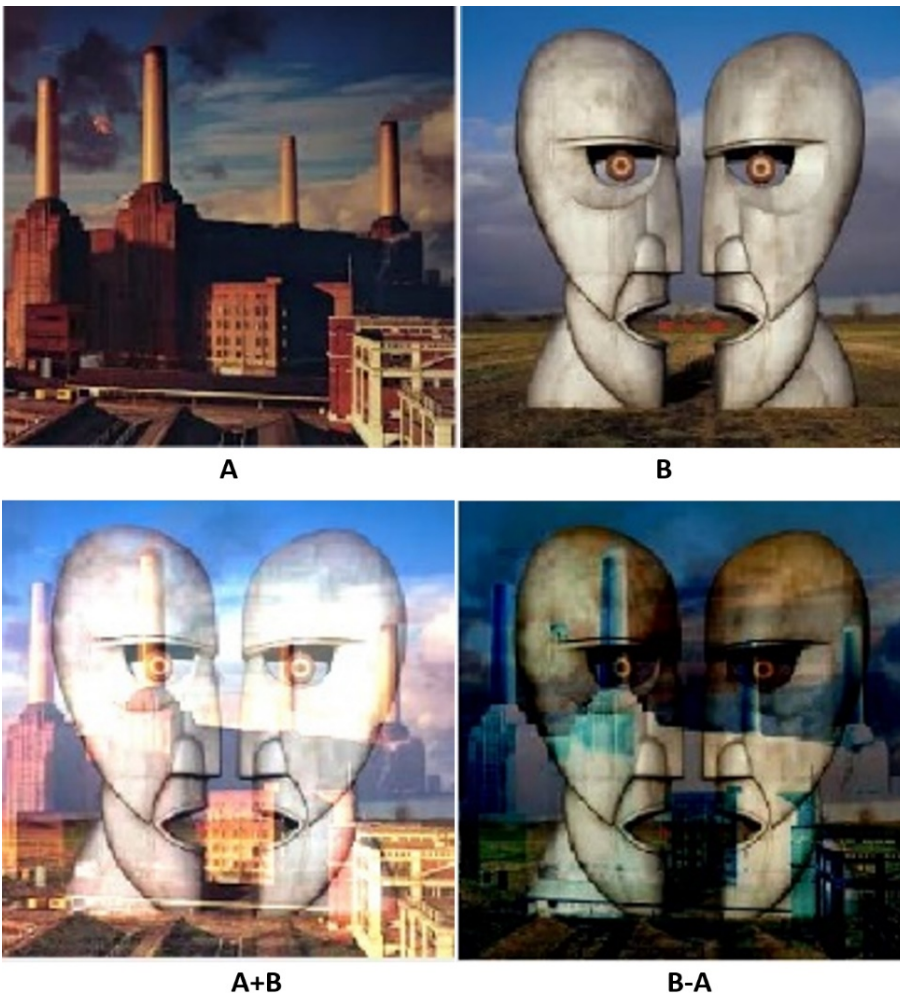
Figure 3. Some matrix operations using digital images (wikipedia.org, n.d.a, n.d.b)

Figure 4. Example of Linear combinatios of two different imagens (Pinterest, n.d.)

Matrix operations can be performed in an interactive user environment and in an intuitive way. Thus, students were asked to create a Graphical User Interface (GUI) in Matlab that was developed to run on any computer or on a web page, such as the LA-CU page. In **Figure 5** are presented two of these user-friendly interfaces that make it easier to operate and learn matrix operations. The user can choose the images (matrices) to operate and can, with just one click, perform the matrix operations as desired.
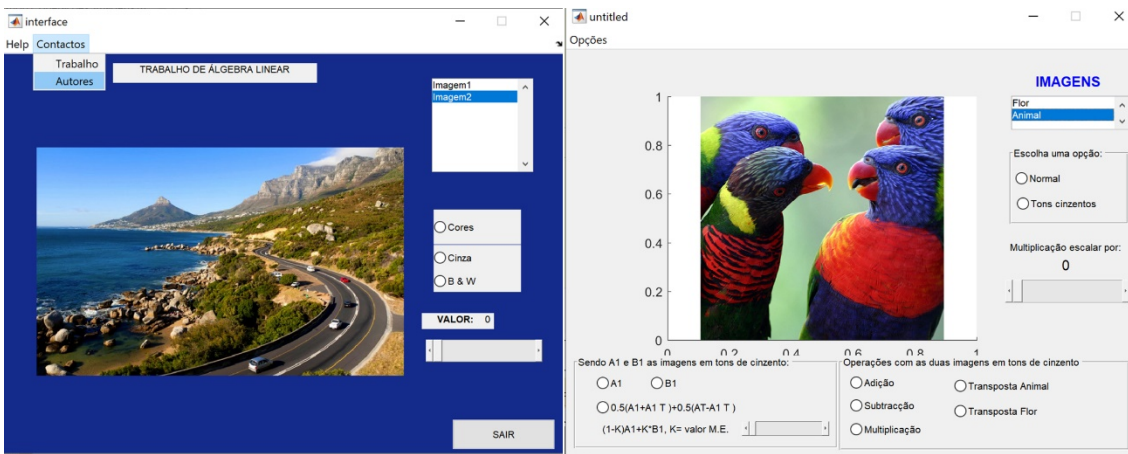
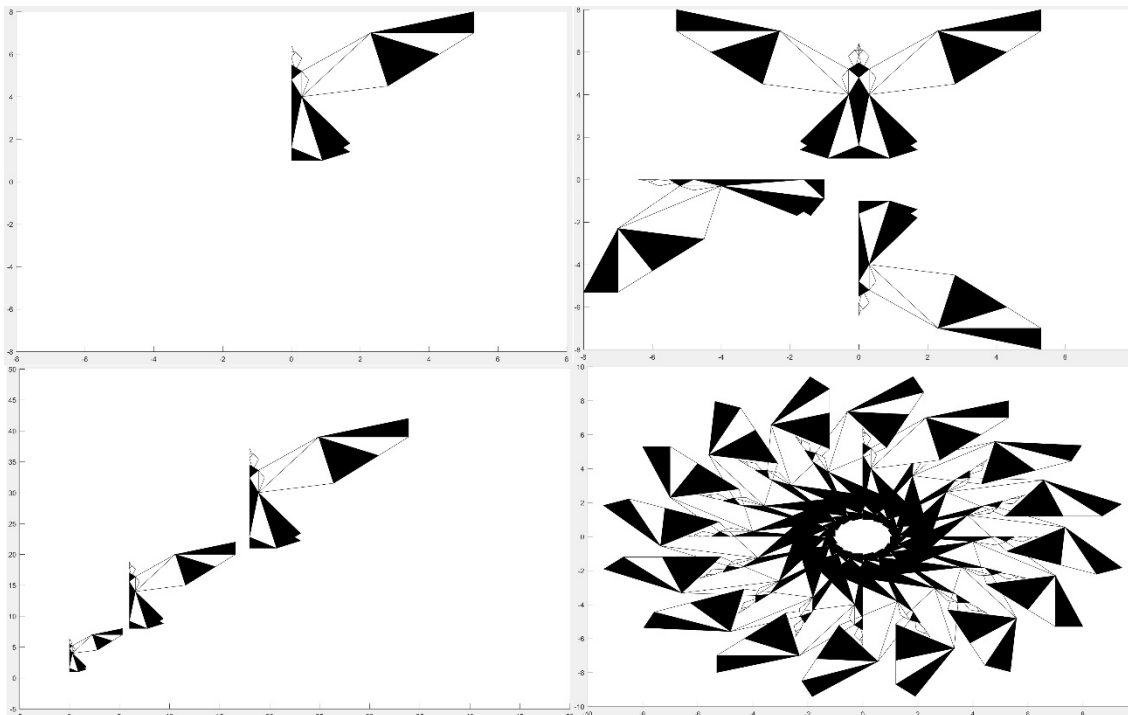**Figure 5.** Graphical interfaces developed in Matlab

Figure 6. Geometric transformation: polygons image (a - top left), refletions (b - top right), scale change (c - bottom left) and rotation (d - bottom right)

A 2D vector could be represented by an ordered pair (x,y) but it could also be represented by a line matrix $[x \quad y]$. Polygons could also be represented in matrix form, by simply place all the coordinates of the vertices into one matrix. The Geometric transformations such as reflection, scale change and rotation, can be performed by multiplying the vertices of the polygons by the transformation matrix. For example a square with the following vertices (1,1), (-1,1), (-1,-1) and (1,-1) can be described by the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \\ 1 & -1 \end{bmatrix}$ and to make the reflection in the x-axis is necessary to multiply this matrix by the reflection matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The example shown in **Figure 6** is a set of polygons that define polygonal areas which are filled or unfilled to simulate the left part of a butterfly (**Figure 6a**)). These sets of polygons can be reflective around the x-axis, the y-axis, and the y = -x line by obtaining the images on the bottom right, top left, and bottom left of **Figure 6b** respectively. Other geometric transformations can be applied to the image in **Figure 6a**) such as scaling (**Figure 6c**) and rotation (**Figure 6d**).

Sam Loyd perhaps America's greatest ever puzzle maker, invented and refined thousands of puzzles in his lifetime (Mathsisfun.com, n.d.). One of the puzzles invented by Sam was the Sliding Block Puzzle and the original version was made with 15 wooden blocks in a tray. A block or submatrix of a matrix A is any matrix obtained from A by deleting rows and columns. The image (matrix) chosen by the students is divided into matrix blocks (sub-

Figure 7. Sliding Block Puzzle image using 16 (left) and 64 (right) blocks (Notícias ao Minuto, n.d.; wikipedia.org, n.d.c)

Figure 8. Posters developed by the students

matrices) that are randomly mixed to simulate a Sliding Block Puzzle. Then one of the blocks, also randomly, is filled with black to simulate the empty block of the puzzle. **Figure 7** shows two simulations of the puzzle. The first puzzle (left) consists of 16 (4x4 sub-matrices) blocks is an image of a traditional Coimbra's guitar and the second by 64 blocks (8x8 sub-matrices) is the image of the Coimbra's University.

These are some applications with DIP using LA developed in practical projects by the engineering students. Other ones can be seen in previous publications (Caridade, 2019, 2011; Caridade et al., 2014).

Students are encouraged to share their work as a scientific communication through poster or oral presentation. The poster is a way for the student to socialize and draw attention to the work done, allowing a more personal, time-free iteration with their peers or teacher. The main challenge for the student who chooses this form of communication is to attract colleagues. In the **Figure 8** you can see two examples of posters made by the students.

Oral presentations have limited time and include questions from peers or the teacher. In this type of communication, the student will have a wide audience and a favorable environment for discussion and analysis of the work presented. In any of the modalities the teacher has a guiding role.

MONITORING AND EVALUATION

The development of a new modality of formation needs to be monitored and evaluated so that the consolidation of practices takes place in a solid and consistent way (Gomes, 2009). There is already some information on higher education assessments, such as Caridade (2011), Ibrahim et al. (2012), Costică (2014) which shows that the application of new technologies to learning mathematics has a positive effect in reducing dropout rates. In addition, Garg and Kaundal (2017) also evaluated students' interest in using LA in computer vision because they provide more truthful and close solution to the real-world problems. The references Zin et al. (2015), Ohrstrom et al. (2005) revealing that students seem to appreciate the contribution of these transforming teaching

Table 2. Profile of CIE students who did the practical work in the 2018/2019 school year

Students profile	Description
Students	1st Year of Mechanical and Electromechanical Engineering
Nº of Students	58 (57 males and 1 females)
Mean of ages	20.55 (ages between 18 and 35 years)
Residence	34.5% Coimbra and 83% of the districts Coimbra, Aveiro, Leiria, Viseu, Santarém.
Work situation	15.5% of students are student workers.
Course	24% Electromechanical Engineering; 76% Mechanical Engineering
Year of admission to the course	57% in 2018 and 43% in previous years.

and learning approach to their lectures. With these experiences enables the evaluation of several cross skills and competencies such as critical thinking and ethical leadership as Izquierdo says (Izquierdo et al., 2016).

This project has been monitoring and evaluated in a systematic way. The assess to teaching and learning of the students and theirs interest and satisfaction are one of the main points to the developed of the practical project proposed. In this sense, a questionnaire was made in Moodle platform to evaluate the importance in developing, projects like this one, by engineering students, in motivating and learning LA contents.

DESCRIPTION AND RESULTS OF THE STUDY

In Polytechnic of Coimbra-Institute of Engineering, since 2016, in LA-CU of Electromechanical and Mechanical Engineering, the students develop practical projects as a complement to the classes. In first semester of 2018-19, 58 students answered a questionnaire to evaluate students' motivation and interest in doing the practical project. The questionnaire was designed to identify the type of student and their opinion regarding the course, the CU and the practical project (Caridade, 2019). Data were processed using SPSS software version 21.

Characteristics of the Students

The student characterization is represented in **Table 2**. From the 58 students, one is female, with a mean age of 21 years, 15.5% of student are student workers, mostly residence outside Coimbra city (83%), 44 (76%) are in Mechanical Engineering and 14 (24%) in Electromechanical Engineering and a total of 43% students who have not completed LA in previous years.

The students involved in this study are first- and second-year students, quite new, who had no contact with image processing and who had very little contact with Matlab software. Their experience with this type of project is practically none. Therefore, some students may not be motivated to do the practical project because they have little time available (for example student workers) to learn how to use Matlab and prepare the requested project (Caridade, 2019).

Choice of Course

The reasons that lead students to choose a school or course are factors that condition their commitment. The quality of education is the most frequent choice for PC-IE (46%), followed by the recommendation of friends (11%). That is, almost half of the students are enrolled in PC-IE for the quality of teaching. The main motivation to attend the degree was the identification with the course (60%) and then the opportunities offered by the course (29%). The degree of satisfaction they have in relation to the chosen course is satisfactory (50%), very satisfactory (38%) and only 12% indicates a regular satisfaction. That is, more than 88% of students are satisfied with the course they have chosen (Caridade, 2019).

Functioning and Organization of the CU

To evaluate the quality of the CU a Likert-type scale was used, with a degree of agreement ranging from 1 to 5, as follows: 1- Very unsatisfactory, 2- Unsatisfactory, 3- Regular, 4- Satisfactory, 5- Very satisfactory. The students interviewed do not attend in the same classes do not all have the same teachers. Students on average are satisfied with the course (28 Satisfactory, 22 Very satisfactory and 6 Regular). **Table 3** shows the mean and standard deviation of the responses obtained to the 6 questions in relation to the quality of LA-CU.

Few students identified regular or unsatisfactory aspects regarding the quality of teaching of CU (DP- 10% regular, RTP- 10% regular and 5% unsatisfactory, CC- 15% regular, TRE- 12% regular, PIU and A- 16% regular and 2% unsatisfactory). The remaining students answered positively to the quality of CU with an average of 4 (Satisfactory) which demonstrates the relevance and interest of the activities involved in CU.

The evaluation of the teacher's role in motivation, explanation and support throughout the teaching and learning process was assessed by 4 questions with the same kind of answers as the previous questions. The answers obtained (**Table 3**-last 4 lines) were only regular, satisfactory and very satisfactory, and the average in the 4 answers

Table 3. Evaluation of the functioning and organization of the CU

Question	Median	Standard deviation
Didactic processes (DP)	4.07	0.525
Relationship between theory and practice (RTP)	4.02	0.737
Curricular Contents (CC)	3.97	0.529
Teaching, research and extension (TRE)	4.02	0.513
Pedagogical instruments used (PIU)	4.00	0.649
Assessment (A)	4.03	0.674
Relationship student teacher	4.59	0.622
Attention of the teacher to doubts	4.60	0.591
Teacher ethics	4.52	0.569
Attendance of the teacher	4.59	0.622

Table 4. Interest and motivation of practical project

Consolidation of learning	Average	Standard deviation
Matrices	4,03	0,556
Types of matrices	4,02	0,574
Dimension of matrices	3,97	0,642
Operations with matrices	4,03	0,668
Properties of operations with matrices	4,59	0,558

is very satisfactory. It was found that less than 5 students answered "regularly" in the 4 questions, so it can be concluded that students like and agree with the way the teacher is available and supports them.

Interest and Motivation of Practical Project

Some students' knowledge of Matlab software give a make difference in the calculus of the estimated time to complete the project. So, it is necessary 2 to 3 days for 17 students (29%), 1 week for 22 students (38%), 2 weeks for 12 students (21%) and more than 3 weeks for 7 students (12%).

Almost all students agree that the Matlab project is interesting (100%), useful (88.7%) and allows motivating learning (90.9%) of the matrix calculation. 4 new questions were elaborated to evaluate the dedication and commitment in terms of the learning and consolidation of the contents covered in the CU. The evaluation was done according to a Likert scale by choosing one of the 5 options: 5-I totally agree, 4 - I agree, 3 - Neither agree nor disagree, 2 - Disagree and 1 - I totally disagree. In **Table 4** it can be see that students agree that the development of the practical project allows the consolidation of their knowledge in matrices, types and dimensions of matrices as well as in operations and properties between matrices (Caridade, 2019).

In an open answer, the students were asked what they liked most during the execution of the developed project. The most significant responses were as follows:

- Motivation for using new tools.
- Explore content differently.
- Manipulation of Images as Matrices.
- Further clarification of content.

Regarding what the students disliked, the answers focused on the time spent and the errors that arise during the project. Words like "interesting", "motivating", "experience", "knowledge" and "consolidate" are most frequencies and indicate a positive opinion from the students about the practical project. To see the word's cloud obtained to this question see the previous paper (Caridade, 2019).

CONCLUSIONS AND FUTURE WORK

The development of the interconnection between DIP and LA arises from the interest in promoting a learning of the LA contents in a motivating way to the Coimbra Institute of Engineering students. In this way, it intends to improve the teaching and learning practices of the LA for students of Mechanical and Electromechanical Engineering of Coimbra Institute of Engineering and possibly the extension to other courses.

Although the experience has been limited to some LA contents, the importance of this type of methodology justifies its presentation. By the way, based on the student's opinion, it is evident that learning is more exciting when using a different methodology that can motivate the interest of the students. This can be seen by the feedback from students. Most of the students enjoy it and they were very active (almost all the enquiry students). This methodology proves to be a viable and effective way of teaching and learning LA. More activities are needed to learn LA concepts more consistently by applying practical results to different images.

In this school year the project already includes other IP activities applied to LA such as simple object counting. The student will have to create an image consisting of different geometric figures of different colors and develop an application that automatically counts the total number of objects as well as the number of objects of a given color. Other activities including LA content related to eigenvalues and eigenvectors are also being considered.

It is also intended to apply the evaluation model described in Machado and Gomes (2013) using the dimensions: subject (students and teachers); pedagogical and organizational and technology (support technologies and support structures and services), to identify different factors to consider in the evaluation. It is also suggested as a future work, that a satisfaction questionnaire be made annually to students and teachers as a means of quantifying the satisfaction of all those involved in the teaching and learning process, identifying problems and making the necessary modifications for the good operation of the practical project.

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