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Convergence Theorems of Nonself Nearly Asymptotically Nonexpansive Mappings for Common Fixed Points in Banach Spaces

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ABSTRACT

Solving problems through fixed point is a powerful tool in mathematics. In this paper we study convergence theorems for two nonself nearly asymptotically nonexpansive mappings. Specifically, we take uniformly convex Banach space for two nonself mappings and an iteration process to find convergence of common fixed point. In this process we prove approximate fixed point results. We also prove strongly convergence theorems for common fixed point and weakly convergence theorem for common fixed point. The research probably involves designing specific iterative schemes that generates sequences that converges to the common fixed point. Analysing the convergence of these schemes is a crucial part of the work. This study contributes to the broader field of fixed point theory by extending existing results to a more general class of mappings, offering insights into the behaviour of nearly asymptotically nonexpansive mappings in the context of common fixed point problems.

Key words- Banach space, fixed point, nearly asymptotically nonexpansive mappings, convergence theorems.

I. INTRODUCTION

The theory of fixed points has seen significant generalizations over time, beginning with foundational work on nonexpansive mappings in uniformly convex Banach spaces. Clarkson [1] introduced the concept of uniform convexity, which plays a critical role in ensuring convergence of iterative sequences. Browder [2] and Opial [3] extended this framework by proving existence and convergence results for nonexpansive mappings. Bose [4] took a further step by studying weak convergence for asymptotically nonexpansive maps, where the nonexpansiveness condition is relaxed over iterations. These developments laid the foundation for more flexible mapping classes, such as asymptotically and nearly asymptotically nonexpansive mappings, which allow for broader applications in nonlinear analysis.

Recent studies have focused on refining these generalizations. Alfuraidan and Khamsi [5] examined monotone asymptotically nonexpansive mappings, extending fixed point results to ordered Banach spaces. Aggarwal, Uddin, and Nieto [6] studied nearly asymptotically nonexpansive mappings and established fixed point theorems under weaker assumptions, demonstrating their convergence behavior even when mappings deviate slightly from nonexpansiveness. Iterative methods for common fixed points have also evolved, with contributions from Akbulut et al. [7], Guo et al. [8], and Khan et al. [9], who explored various types of asymptotically nonexpansive and nonself mappings. These advancements collectively illustrate the flexibility and strength of nearly asymptotically nonexpansive mappings in addressing complex problems in fixed point theory.

Based on above literature the authors found following gaps: convergence for fixed point of two non self nearly asymptotically nonexpensive mappings has not been considered yet. To fill the identified gap in the literature we

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have proven the convergence of the fixed point of two non self nearly asymptotically non expensive mappings. Further, authors have extended this and included the strong and weak convergence for the work.

Let E be any Banach space whose convex closed subset of K is not empty. The nonexpansive retraction of E to K is denoted by $R: E \to K$. A point $a \in K$ where $T(a) = a$ is a nonself mapping $T: K \to E$ with a fixed point of T. Furthermore, the $F(T)$ represents the set of all fixed points of T. The existence theorems of fixed points for asymptotically nonexpansive mappings have been studied by a few authors [10, 11, 12, 13].

Given a Banach space E *and* a nonempty subset C, fix sequence $\{a_n\}$ in $[0, \infty]$ with $a_n \to 0$. If, for every n *in* $\mathbb{N}, \exists$ a constant $k_n \geq 0$ such that, for $\{a_n\}$, a mapping $T: C \to C$ is nearly Lipschitzian

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \ for\ all\ x, y \in C. \qquad (1.1)$$

The nearly Lipschitz constant, represented as $\eta(T^n)$, is the minimum of constants k_n for which (1.1) is holds. Observe that

$$\eta(T^n) = sup\left\{\frac{\|T^n x - T^n y\|}{\|x - y\| + a_n}: x, y \in C, x \neq y\right\}.$$

A nearly Lipschitzian map T with sequence $\{(a_n, \eta(T^n))\}$ is called nearly asymptotically nonexpansive [14], if $\eta(T^n) \geq 1, \forall\ n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \eta(T^n) \leq 1$. Sahu [14] in 2005, established the class of nearly asymptotically nonexpansive mapping.

Understanding the geometry of spaces, sequence convergence, and the demiclosed ness principle of nonlinear mappings all depend on the Opial's condition. The Opial's condition is satisfied by any space X if a sequence $\{\gamma_n\}$ defined on space X converges weakly to any $\gamma_0 \in X$ then

$$\lim_{n\to\infty} inf\|\gamma_n - \gamma_0\| < \lim_{n\to\infty} inf\|\gamma_n - \gamma\| \ \forall \gamma \in X\ and\ \gamma \neq \gamma_0$$

here, we can establish weak Opial's condition by substituting the inequality $\leq$ for the strict inequality $<$.

Theorem 1. [15] Assume that B is a Hilbert space *and* $G \neq \emptyset$ is convex closed & bounded subset of B. Let T be a mapping of nearly asymptotically nonexpansive from G *to* G with sequence $\{k_n\} \subset [1, \infty)$ *for all* $n \geq 1, \lim_{n\to\infty} = 1$ *and* $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in [0,1] that, for some constant a, b, satisfies the constraint $0 < a \leq \alpha_n \leq b < 1, n \geq 1$. Then the $\{x_n\}$ sequence produced from any $x_1 \in K$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, n \geq 1,$$

strongly convergence to any fixed point of T.

Several theorems of strong *and* weak convergence in convex Banach spaces for asymptotically nonexpansive mappings were developed by Chidume, Ofoedu, and Zegeye [16] in 2003. Theorems of strong *and* weak convergence on convex uniformly Banach spaces and other convex spaces for nonexpansive mappings have been studied by a few authors [13, 17, 18, 19]. Wang [20] discussed additional fixed-point results using asymptotically nonexpansive mapping. John and Shaini recently explored the fixed-point theorems for Suzuki nonexpansive mapping in Banach space [21]. For fixed points of nonself asymptotically nonexpansive mappings, Wei and Jing [22] introduce a novel iteration technique.

Motivated by these concepts, we present an iterative method for evaluating the common fixed points convergence theorem of an $T_1, T_2: K \to E$ be two nonself nearly asymptotically nonexpansive mappings formed on Banach space. The set of common fixed points of an $F(T_1) \cap F(T_2) = \{x \in K: T_1 x = T_2 x = x\} \neq \emptyset$ of nonself nearly asymptotically nonexpansive mappings. A mapping $R: E \to E$ is said to be retraction if $R^2 = R$.

II. PRELIMINARIES

Definition 2.1. [23] A mapping $T: C \to C$ defined on space C is called nonexpansive mapping if $\|T\mu - T\gamma\| \leq \|\mu - \gamma\|, \forall\ \mu, \gamma$ in space C.

Definition 2.2. [14] Let E be a Banach space with nonempty subset K, fix sequence $\{a_n\}$ in $[0, \infty]$ with $a_n \to 0$. Let nonexpansive retraction R from E to K. A map $T: K \to E$ is called nearly Lipschitzian with $\{a_n\}$ if $\forall\ n \in \mathbb{N}, \exists$ *a constant* $k_n \geq 0$ such that

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$$\|T(RT)^{n-1}x - T(RT)^{n-1}y\| \le k_n(\|x-y\| + a_n) \;\; for\ all\ x, y \in K. \qquad (1.2)$$

The nearly Lipschitz constant, represented as $\eta(T^n)$, is the infimum of constants k_n for which (1.2) is holds. Observe that

$$\eta(T^n) = sup\left\{\frac{\|T(RT)^{n-1}x - T(RT)^{n-1}y\|}{\|x-y\| + a_n} : x, y \in C, x \neq y\right\}.$$

A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is called nearly asymptotically nonexpansive, if $\eta(T^n) \ge 1, \forall\, n \in \mathbb{N}\ and\ \lim_{n\to\infty} \eta(T^n) \le 1$.

Result 2.3. [13] Let K be a nonempty convex closed subset of a uniformly convex Banach space with nonexpansive retraction R. Let $T_1, T_2: K \to E$ be mappings that are nonself and nearly asymptotically nonexpansive. The following iteration approach is used to approximate the common fixed points of two nonself nearly asymptotically nonexpansive mappings:

$$\begin{cases} \mu_1 \in K, \\ \mu_{n+1} = R\big((1-p_n)\mu_n + p_n T_1(RT_1)^{n-1}\vartheta_n\big), \\ \vartheta_n = R\big((1-q_n)\mu_n + q_n T_2(RT_2)^{n-1}\mu_n\big), \;\; n \ge 1, \end{cases}$$

where $\{p_n\}$ & $\{q_n\}$ are two sequences in $[0, 1)$.

Lemma 2.4. [24] Let $\{b_n\}, \{\delta_n\}$, and $\{c_n\}$ be sequence of real, nonnegative numbers satisfying

$$b_{n+1} \le (1+\delta_n)b_n + c_n, \forall\, n \ge 1$$

if $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} b_n$ exists.

Lemma 2.5. [25] Let E be a uniformly convex real Banach space, and let a and b be two constants such that $0 < a < b < 1$. Assume that there are two sequences in E, $\{x_n\}, \{y_n\}$ and that $\{t_n\} \subset [a, b]$ is a real sequence. Then

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d, \lim_{n\to\infty}\ sup\|x_n\| \le d, \lim_{n\to\infty}\ sup\|y_n\| \le d$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \ge 0$ is a constant.

Lemma 2.6. [16] Let K be a nonempty closed subset of E, and let E be a real uniformly convex Banach space, and let $T: K \to E$ be a nonself nearly asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)\ and\ k_n \to 1\ as\ n \to \infty$. Then $I - T$ is demiclosed at 0.

III. MAIN RESULTS

Lemma 3.1. Let K be a nonempty closed convex subset of a Banach space E with nonexpansive retraction R. Let $T_1, T_2: K \to E$ be nonself nearly asymptotically nonexpansive mappings with corresponding sequences $\{k_n\} \subset [1, \infty)\ and\ \{a_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $\sum_{n=1}^{\infty} a_n(k_n + 1) < \infty$. Suppose that $\{\mu_n\}$ is defined by Result 2.3, where $\{p_n\},\ \{q_n\} \in [0, 1)$. If $F(T_1) \cap F(T_2) \neq \emptyset$ then $\lim_{n\to\infty} \|\mu_n - \rho\|$ exists for each fixed point $\rho \in F(T_1) \cap F(T_2)$.

Proof: For any $\rho \in F(T_1) \cap F(T_2)$ by Result 2.3, we have

$$\begin{aligned} \|\mu_{n+1} - \rho\| &\le \|(1-p_n)(\mu_n - \rho) + p_n(T_1(RT_1)^{n-1}\vartheta_n - \rho)\| \\ &\le (1-p_n)\|\mu_n - \rho\| + p_n[k_n\|\vartheta_n - \rho\| + a_n] \end{aligned}$$

where

$$\begin{aligned} \|\vartheta_n - \rho\| &\le \|(1-q_n)(\mu_n - \rho) + q_n(T_2(RT_2)^{n-1}\mu_n - \rho)\| \\ &\le (1-q_n)\|\mu_n - \rho\| + q_n[k_n\|\mu_n - \rho\| + a_n] \\ &\le (1 - q_n + q_n k_n)\|\mu_n - \rho\| + q_n a_n \end{aligned}$$

thus

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$$\|\mu_{n+1}-\rho\| \le (1-p_n)\|\mu_n-\rho\| + p_n[k_n\|\vartheta_n-\rho\|+a_n]$$
$$\le (1-p_n)\|\mu_n-\rho\| + p_n[k_n(1-q_n+q_nk_n)\|\mu_n-\rho\|+k_nq_na_n]$$
$$\le [(1-p_n)+p_nk_n(1-q_n+q_nk_n)]\|\mu_n-\rho\| + p_nk_nq_na_n+p_na_n$$

now

$$\delta_n = -p_n + p_nk_n(1-q_n+q_nk_n)$$
$$= p_n[(-1+k_n)+q_nk_n(-1+k_n)]$$
$$= p_n(-1+k_n)(1+k_nq_n)$$
$$\le p_n(k_n-1)(1+k_n)$$
$$\le (k_n^2-1)$$

because

$$1+k_nq_n \le 1+k_n, \qquad p_n, q_n \in [0,1)$$

and

$$b_n = \|\mu_n-\rho\|,$$
$$c_n = p_nk_nq_na_n + p_na_n$$
$$\le k_na_n + a_n$$
$$= a_n(k_n+1)$$

from Lemma 2.4, we know that if

$$b_{n+1} \le (1+\delta_n)b_n + c_n, \forall\, n \in \mathbb{N}$$

such that $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}c_n < \infty$ then $\lim_{n\to\infty} b_n$ exists so, we get $\lim_{n\to\infty}\|\mu_n-\rho\|$ exists.

Lemma 3.2 Let K be a nonempty closed convex subset of a Banach space E with nonexpansive retraction R. Let $T_1, T_2: K \to E$ be nonself nearly asymptotically nonexpansive mapping with corresponding sequences $\{k_n\} \subset [1,\infty)$ and $\{a_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^2-1) < \infty$ and $\sum_{n=1}^{\infty}a_n(k_n+1) < \infty$. Suppose that $\{\mu_n\}$ is defined by Result 2.3, where $\{p_n\}$, $\{q_n\} \in [0,1)$. If $F(T_1)\cap F(T_2) \ne \emptyset$ then $\lim_{n\to\infty}\|\mu_n - T_1\mu_n\| = \lim_{n\to\infty}\|\mu_n - T_2\mu_n\| = 0$.

Proof. Taking $\rho \in F(T_1)\cap F(T_2)$, by Lemma 3.1, we see that $\lim_{n\to\infty}\|\mu_n-\rho\|$ exists. Let $\lim_{n\to\infty}\|\mu_n-\rho\| = c$. From Result 2.3, we have

$$\|\vartheta_n-\rho\| \le (1-q_n+q_nk_n)\|\mu_n-\rho\| + q_na_n \;\ldots..(A)$$

lim *sup* taking both the sides in (A), we obtain

$$\lim_{n\to\infty}\sup\|\vartheta_n-\rho\| \le \lim_{n\to\infty}\|\mu_n-\rho\| \le \mathrm{c} \;\ldots..(B)$$

In addition,

$$\|T_1(RT_1)^{n-1}\vartheta_n-\rho\| \le k_n\|\vartheta_n-\rho\| + \mathrm{a_n}$$

lim *sup* taking both the sides in this above inequality, we get

$$\lim_{n\to\infty}\sup\|T_1(RT_1)^{n-1}\vartheta_n-\rho\| \le c \;\ldots..(C)$$

since

$$\lim_{n\to\infty}\|\mu_n-\rho\| = \mathrm{c}$$

then

$$\lim_{n\to\infty}\|(1-p_n)(\mu_n-\rho) + p_n(T_1(RT_1)^{n-1}\vartheta_n-\rho)\| = c$$

by Lemma 2.5, we have

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$$\lim_{n\to\infty}\|\mu_n - T_1(RT_1)^{n-1}\vartheta_n\| = 0 \ldots..(D)$$

in addition,

$$\|\mu_n - \rho\| \le \|\mu_n - T_1(RT_1)^{n-1}\vartheta_n\| + \|T_1(RT_1)^{n-1}\vartheta_n - \rho\|$$
$$\le \|\mu_n - T_1(RT_1)^{n-1}\vartheta_n\| + k_n\|\vartheta_n - \rho\| + \mathrm{a_n}$$

lim inf taking both the sides in above inequality, by (D), we obtain

$$\lim_{n\to\infty}\inf\|\vartheta_n - \rho\| \ge \mathrm{c} \ldots..(E)$$

now, by (B) and (E) we get $\lim_{n\to\infty}\|\vartheta_n - \rho\| = \mathrm{c}$. It implies that

$$\lim_{n\to\infty}\|(1-q_n)(\mu_n - \rho) + q_n(T_2(RT_2)^{n-1}\mu_n - \rho)\| = c$$

by Lemma 2.5,

$$\lim_{n\to\infty}\|\mu_n - T_2(RT_2)^{n-1}\mu_n\| = 0 \ldots..(F)$$

now by Result 2.5, we get

$$\lim_{n\to\infty}\|\vartheta_n - T_2(RT_2)^{n-1}\mu_n\| = 0 \ldots..(G)$$

we now prove that

$$\lim_{n\to\infty}\|\mu_n - T_1(RT_1)^{n-1}\mu_n\| = 0$$

$$\|\mu_n - T_1(RT_1)^{n-1}\mu_n\| \le \|\mu_n - T_1(RT_1)^{n-1}\vartheta_n + T_1(RT_1)^{n-1}\vartheta_n - T_1(RT_1)^{n-1}\mu_n\|$$
$$\le \|\mu_n - T_1(RT_1)^{n-1}\vartheta_n\| + k_n\|\mu_n - \vartheta_n\| + a_n$$
$$\le \|\mu_n - T_1(RT_1)^{n-1}\vartheta_n\| + k_n\|q_n(\mu_n - T_2(RT_2)^{n-1}\mu_n)\| + a_n$$

thus by (D) and (F), we get

$$\lim_{n\to\infty}\|\mu_n - T_1(RT_1)^{n-1}\mu_n\| = 0 \ldots..(H)$$

now by (F) and (H), we get

$$\lim_{n\to\infty}\|T_1(RT_1)^{n-1}\mu_n - T_2(RT_2)^{n-1}\mu_n\| = 0 \ldots..(I)$$

by (D) $and\ Result$ 2.5, we get

$$\lim\|\mu_{n+1} - T_1(RT_1)^{n-1}_{n\to\infty}\vartheta_n\| = 0 \ldots..(J)$$

Hence

$$\|\mu_n - T_1\mu_n\| = \|\mu_n - T_1(RT_1)^{n-1}\mu_n + T_1(RT_1)^{n-1}\mu_n - T_1\mu_n\|$$
$$\le \|\mu_n - T_1(RT_1)^{n-1}\mu_n\| + \|T_1(RT_1)^{n-1}\mu_n - T_1(RT_1)^{n-1}\vartheta_{n-1}\| + \|T_1(RT_1)^{n-1}\vartheta_{n-1} - T_1\mu_n\|$$
$$\le \|\mu_n - T_1(RT_1)^{n-1}\mu_n\| + k_n\|\mu_n - \vartheta_{n-1}\| + a_n + k_n\|T_1(RT_1)^{n-2}\vartheta_{n-1} - \mu_n\| + a_n \ldots.(K)$$

from (J) that

$$\lim_{n\to\infty}\|T_1(RT_1)^{n-2}\vartheta_{n-1} - \mu_n\| = 0 \ldots.(M)$$

in addition,

$$\|\mu_{n+1} - \vartheta_n\| = \|\mu_{n+1} - T_1(RT_1)^{n-1}\vartheta_n + T_1(RT_1)^{n-1}\vartheta_n - \vartheta_n\|$$
$$\le \|\mu_{n+1} - T_1(RT_1)^{n-1}\vartheta_n\| + \|\vartheta_n - \mu_n\| + \|\mu_n - T_1(RT_1)^{n-1}\vartheta_n\|$$
$$\le \|\mu_{n+1} - T_1(RT_1)^{n-1}\vartheta_n\| + q_n\|T_2(RT_2)^{n-1}\mu_n - \mu_n\| + \|\mu_n - T_1(RT_1)^{n-1}\vartheta_n\|$$

using $(D), (F), (J)$ and $(K), we\ get$

$$\lim_{n\to\infty}\|\mu_{n+1}-\vartheta_n\| = 0 \ldots..(N)$$

by $(H), (F), and\ (N)$, it follows from$(K)\ that$

$$\lim_{n\to\infty}\|\mu_n - T_1\mu_n\| = 0$$

similarly, we may show that

$$\lim_{n\to\infty}\|\mu_n - T_2\mu_n\| = 0.$$

Theorem 3.3. Let K be a nonempty closed convex subset of a Banach space E with nonexpansive retraction R. Let $T_1, T_2: K \to E$ be nonself nearly asymptotically nonexpansive mapping with corresponding sequences $\{k_n\} \subset [1,\infty)\ and\ \{a_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $\sum_{n=1}^{\infty} a_n(k_n + 1) < \infty$. Suppose that $\{\mu_n\}$ is defined by Result 2.3, where $\{p_n\},\ \{q_n\} \in [0,1)$. If either $T_1\ or\ T_2$ is completely continuous, $and\ F(T_1) \cap F(T_2) \neq \emptyset$ then $\{\mu_n\}$ strongly converges to a common fixed point of $T_1\ and\ T_2$.

Proof. By Lemma 3.1, $\{\mu_n\}$ is bounded and by Lemma 3.2,

$$\lim_{n\to\infty}\|\mu_n - T_1\mu_n\| = 0$$

and

$$\lim_{n\to\infty}\|\mu_n - T_2\mu_n\| = 0$$

then $\{T_1\mu_n\}, \{T_2\mu_n\}\ are$ bounded. If T_1 continuous completely, ∃ subsequence $\left\{T_1\mu_{n_j}\right\}\ of\ \{T_1\mu_n\}$ such that $T_1\mu_{n_j} \to \rho\ as\ j \to \infty$.

It follows from Lemma 3.2 that

$$\lim_{j\to\infty}\left\|\mu_{n_j} - T_1\mu_{n_j}\right\| = \lim_{j\to\infty}\left\|\mu_{n_j} - T_2\mu_{n_j}\right\| = 0.$$

So by the continuity of T_1 and Lemma 2.6, we have

$$\lim_{j\to\infty}\left\|\mu_{n_j} - \rho\right\| = 0\ and\ \rho \in F(T_1) \cap F(T_2).$$

Then by Lemma 3.1, we get

$$\lim_{j\to\infty}\|\mu_n - \rho\|\ exists.$$

Thus

$$\lim_{j\to\infty}\|\mu_n - \rho\| = 0.$$

Theorem 3.4. Let K be a nonempty closed convex subset of a Banach space E with nonexpansive retraction R. Let $T_1, T_2: K \to E$ be nonself nearly asymptotically nonexpansive mapping with corresponding sequences $\{k_n\} \subset [1,\infty)$ and $\{a_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $\sum_{n=1}^{\infty} a_n(k_n + 1) < \infty$. Assume $\{\mu_n\}$ is defined by Result 2.3, where $\{p_n\},\ \{q_n\} \in [0,1)$. If either $T_1\ or\ T_2$ is demicompact, and $F(T_1) \cap F(T_2) \neq \emptyset$ then $\{\mu_n\}$ converges strongly to a common fixed point of $T_1\ and\ T_2$.

Proof. Let T_1 and T_2 is demicompact, $\{\mu_n\}$ is bounded and

$$\lim_{n\to\infty}\|\mu_n - T_1\mu_n\| = \lim_{n\to\infty}\|\mu_n - T_2\mu_n\| = 0,$$

then ∃ $\left\{\mu_{n_j}\right\}$ subsequence of $\{\mu_n\}$ such that $\left\{\mu_{n_j}\right\}$ strongly converges to ρ. It is inferred from Lemma 3.1. Given that $\left\{\mu_{n_j}\right\}$ subsequence of $\{\mu_n\}$ such that $\left\{\mu_{n_j}\right\}$ strongly convergences to ρ then $\{\mu_n\}$ strongly converges to the common fixed point $\rho \in\ F(T_1) \cap F(T_2) \neq \emptyset$.

Theorem 3.5. Let K be a nonempty convex closed subset of a uniformly convex Banach space E satisfying Opial's condition. Let $T_1, T_2: K \to E$ be nonself nearly asymptotically nonexpansive mapping with corresponding sequences $\{k_n\} \subset [1,\infty)$ and $\{a_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $\sum_{n=1}^{\infty} a_n(k_n + 1) < \infty$. Assume that $\{\mu_n\}$ is defined by

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Result 2.3, where $\{p_n\}$, $\{q_n\} \in [0,1)$. If $F(T_1) \cap F(T_2) \neq \emptyset$ then $\{\mu_n\}$ weakly converges to common fixed point of T_1 *and* T_2.

Proof. For any $\rho \in F(T_1) \cap F(T_2)$ then from Lemma 3.1 $\lim_{n\to\infty} \|\mu_n - \rho\|$ exists. Now we prove $\{\mu_n\}$ has a weak subsequential limit unique in $F(T_1) \cap F(T_2)$. Now, let ρ_1 *and* ρ_2 be weak limits of subsequence's $\{\mu_{n_k}\}$ *and* $\{\mu_{n_j}\}$ *of* $\{\mu_n\}$. By Lemma 3.2 and 2.6, we know $\rho_1, \rho_2 \in F(T_1) \cap F(T_2)$. Now let $\rho_1 \neq \rho_2$ then by Opial's condition, we get

$$\lim_{n\to\infty} \|\mu_n - \rho_1\| = \lim_{k\to\infty} \|\mu_{n_k} - \rho_1\| < \lim_{k\to\infty} \|\mu_{n_k} - \rho_2\| = \lim_{j\to\infty} \left\|\mu_{n_j} - \rho_2\right\| < \lim_{k\to\infty} \|\mu_{n_k} - \rho_1\| = \lim_{n\to\infty} \|\mu_n - \rho_1\|,$$

which is a contradiction, hence $\rho_1 = \rho_2$. Then $\{\mu_n\}$ weakly converges to common fixed point of T_1 *and* T_2.

Numerical Example

Let X=R and consider the closed interval $C = [0,2]$. Define the standard metric on X by $d(x,y) = |x-y|$ for all $x, y \in X$.

Now, define a mapping $T: C \to C$ as follows:

$$\mathrm{T(x)} = \begin{cases} 1, & if\ x \in [0,1) \\ 2, & if\ x \in [1,2] \end{cases}$$

It can be verified that the function T is nearly asymptotically nonexpansive, and the point $x = 2$ serves as a fixed point of this mapping.

To illustrate convergence, consider an initial value such as $x = 0$ or $x = 0.5$. Through successive iterations under T, the sequence generated by the mapping converges to the fixed point 2.

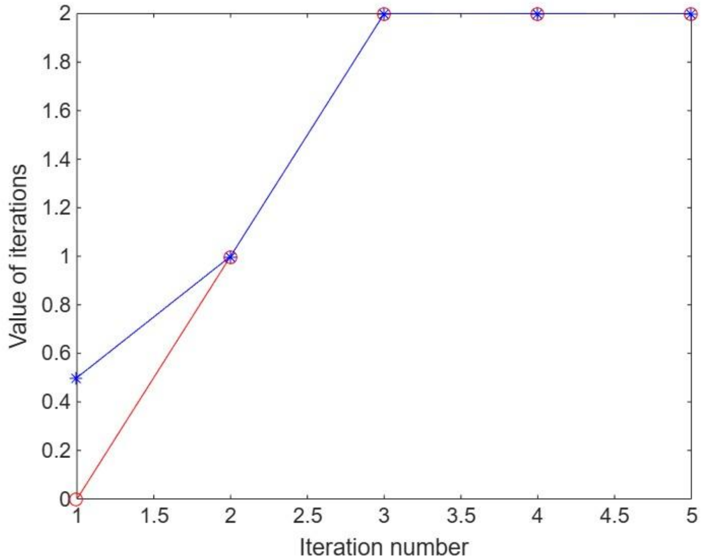

Figure 1. Convergence behaviour

Table 1. Convergence table of Explicit iteration for initial point as $x = 0$ or $x = 0.5$

Explicit iteration number	$x(0) = 0$	$x(0) = 0.5$
1.	0.0000	0.5000
2.	1.0000	1.0000
3.	2.0000	2.0000
4.	2.0000	2.0000
5.	2.0000	2.0000

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CONCLUSION

In this research, we have discussed the iterative scheme for identifying the convergence of fixed-points of nearly asymptotically nonexpansive mapping defined on Banach space. We obtained some convergence results concerning the exhibition of iterative method obtained by satisfy the explicit iteration scheme

$$\begin{cases} \mu_1 \in K, \\ \mu_{n+1} = R\big((1-p_n)\mu_n + p_n T_1 (RT_1)^{n-1}\vartheta_n\big), \\ \vartheta_n = R\big((1-q_n)\mu_n + q_n T_2 (RT_2)^{n-1}\mu_n\big), \ \ n \geq 1, \end{cases}$$

on the nearly asymptotically nonexpansive mapping on Banach space.

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