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# A Novel Survival Class NBULC of Life Distributions: Statistical Testing and Applications to Scientific Data

Silvana T. Gerges [1], E. S. El-Atfy [2], S. E. Abu-Youssef [3]
[1] *Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt*
[2] *Department of Mathematics, Faculty of Science for (girls), Al-Azhar University, Nasr City, Egypt*
[3] *Department of Mathematics, Faculty of Science for (boys), Al-Azhar University, Nasr City, Egypt*

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**ABSTRACT**

This paper discusses a novel approach by examining the failure patterns in the recorded survival data. Following the use of the suggested strategy, survival data are recorded. It is examined to be based on a constant failure rate (null hypothesis) or the new class called new better than used in the Laplace transform of convex order (NBULC) class of life distribution (alternative hypothesis); then the data in use provided a better or higher total present value than an older component in convex (positive or negative effects). The proposed class of life distribution NBULC included many classes, like NBU (new better than used) and NBUL (new better than in Laplace transform) classes of life distributions. The suggested test statistics for our class of life distribution is based on the goodness-of-fit method for non-censored and censored samples. The distribution of this test statistic is investigated via a simulation study. Scientific data is considered an application that utilizes real test data.

**Keywords:** NBULC class, Goodness-of-fit methodology, Pitman's asymptotic efficiency, Monte- Carlo null distribution critical points, non-censored and censored data.

AMS subject classification: 60K10, 62E10, 62N05.

## 1. MOTIVATION AND INTRODUCTION

The survival data set's failure behavior includes figuring out whether the failure rate is constant (null hypothesis $H_0$) or decreasing or increasing (alternative hypothesis $H_1$). The exponential test is crucial and has been used to depict events that are not constrained by the statute of limitations in a variety of classes of life distributions. It guarantees that the longevity of the phenomenon is unaffected by its prior duration. The exponential distribution is compared to multiple classes of life distributions. For instance, Gadallah et al. (2022), Bakr et al. (2024), EL-Sagheer et al. (2022) and Mansour (2020).

Bakr et al. (2022), El-Morshedy et al. (2022), and others have provided an exponential test for various classes of life distributions and their applications in different fields of science, including medical, industrial, economic, and life sciences.

The methodology for testing exponentiality, which is based on the goodness of fit, is discussed by Abu-Youssef and Silvana (2022), Quaid et al. (2024), and others.

The new class NBULC of life distribution is defined by the Laplace transforms approach; the Laplace transform order is a mathematical tool that has been extensively examined in the wider context of reliability analysis. This can be done in Denuit (2001), Stoyan and Muller (2002), and Ahmed and Kayid (2004). Also, the NBULC class includes the well-known classes NBU and NBUL of life distributions.

The NBU class of life distribution is expressed through stochastic comparisons between the remaining life of an old item and a new one, resulting in the creation of different classes of life distribution. Among these is the increasing failure rate (IFR). NBU is explored by Bryson and Siddiqui (1969), and Barlow and Proschan (1981).

In this study, the NBULC class of life distribution is given. And then, the novel test statistic using the goodness-of-fit method is proposed for testing exponentiality versus our class NBULC. The proposed test statistics are evaluated

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for efficiency and comparison to other tests. The power of this test is estimated by computed and tabulated critical values. Finally, our test statistic is applied to censored and complete data in medical science.

**Definition (1):**

**(i)** $x \in$ NBU if

$$\bar{F}(t+x) \le \bar{F}(t)\bar{F}(x), \qquad for\ all\ x, t > 0. \tag{1}$$

**(ii)** $x \in$ NBUL if

$$\int_0^\infty e^{-sx}\bar{F}(x+t)dx \le \bar{F}(t)\int_0^\infty e^{-sx}\bar{F}(x)dx \tag{2}$$

Yue and Cao (2001), Gao et al. (2003), Diab (2010) and Bakr et al. (2024) have studied the class NBUL of life distribution. A new class, NBULC of life distributions, is defined as the following:

**Definition (2):** New better than used in the Laplace transform of convex ordering ($x \in$ NBULC) if

$$\int_x^\infty e^{-sy}\bar{F}(y+t)dy \le \bar{F}(t)\int_x^\infty e^{-sy}\bar{F}(y)dy, \qquad s \ge 0 \tag{3}$$

It is cleared that from Eq. (1), (2) and (3) that:

$$NBU \subset NBUL \subset NBULC.$$

Eq. (3) becomes as follows:

$$\int_x^\infty e^{-sy}\gamma(y)dy \le \mu\int_x^\infty e^{-sy}\bar{F}(y)dy, \tag{4}$$

The rest of the paper is arranged as follows: Pitman's asymptotic efficiency is demonstrated in Section (2) along with a novel test statistic by using the goodness-of-fit technique. In Section (3), the power of the new test statistics is computed, and the critical points for the lower and higher percentile values of the suggested test statistics are tabulated. The treatment of censored data is discussed in Section 4. Section 5 presents applications in medical science to assess the effectiveness of our proposed test.

## 2. TESTING EXPONENTIALITY VERSUS NBULC CLASS OF NON-CENSORED DATA

The suggested test statistic is constructed as a U-statistic in this section, along with a discussion of its asymptotic normality. To assess the method's quality, we compare the asymptotic efficiency of two alternatives in the class NBULC of life distributions.

### 2.1. Statistical Test Measures

A goodness of fit approach is employed to test the null hypothesis, $H_0$: F is exponential, against the alternative hypothesis $H_1$: F belongs to the NBULC class but is not exponential."

Based on the sample $X_1, X_1, \dots, X_n$ from a population with the distribution F, according to Eq. (3), the measure of departure from $H_0: F$ is

$$\delta_S = \int_0^\infty \left[\mu\int_x^\infty e^{-su}\bar{F}(u)du - \int_x^\infty e^{-su}\gamma(u)du\right]e^{-x}dx, \tag{5}$$

The following theorem is fundamental for the evolution of our test statistics.

**Theorem 1.** Let X be an NBULC random variable with distribution F, then

$$\delta_s = \frac{1}{1+s}\left(\mu + \frac{1}{1+s}\right)\int_0^\infty e^{-x(1+s)}dF(x) - \frac{1}{s}\left(\mu + \frac{1}{s}\right)\int_0^\infty e^{-sx}dF(x) + \frac{1+2s}{s^2(1+s)^2}, \tag{6}$$

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Where, $\mu = \int_0^\infty \bar{F}(t)dt.$

**Proof:** From Eq. (5), we have

$$\delta_S = \mu \int_0^\infty e^{-x} \int_x^\infty e^{-su}\bar{F}(u)dudx - \int_0^\infty e^{-x} \int_x^\infty e^{-su}\gamma(u)dudx. \quad (7)$$

The first term of Eq. (7) is given by

$$\mu \int_0^\infty e^{-x} \int_x^\infty e^{-su}\bar{F}(u)dudx = \mu \int_0^\infty e^{-x} \left[\frac{1}{s}e^{-sx} - \int_x^\infty e^{-su}F(u)du\right]dx$$

$$= \mu \int_0^\infty e^{-x}\frac{1}{s}\left[e^{-sx}\bar{F}(x) - \int_x^\infty e^{-su}dF(u)\right]dx$$

$$= \frac{\mu}{s}\left[\frac{1}{1+s} + \left(1 - \frac{1}{1+s}\right)\int_0^\infty e^{-x(1+s)}dF(x) - \int_0^\infty e^{-sx}dF(x)\right]. \quad (8)$$

The second term of Eq. (7) is given by

$$\int_0^\infty e^{-x} \int_x^\infty e^{-su}\gamma(u)dudx = \int_0^\infty e^{-su}\gamma(u)du - \int_0^\infty e^{-u(1+s)}\gamma(u)du$$

$$= -\frac{1}{s}\left[-\mu + \frac{1}{s} - \frac{1}{s}\int_0^\infty e^{-su}dF(u)\right] + \frac{1}{1+s}\left[-\mu + \frac{1}{1+s} - \frac{1}{1+s}\int_0^\infty e^{-u(1+s)}dF(u)\right]$$

$$= \frac{\mu}{s(1+s)} - \frac{1+2s}{s^2(1+s)^2} + \frac{1}{s^2}\int_0^\infty e^{-sx}dF(x) - \frac{1}{(1+s)^2}\int_0^\infty e^{-x(1+s)}dF(x). \quad (9)$$

From Eq. (8) and Eq. (9), the result is obtained.

**Note That:**

**(1)** It is clear to see that if F is exponential, then $\delta_s = 0$.
**(2)** Under $H_1$, $\delta_s > 0$.

To estimate $\delta_s$, we use a random sample $X_1, X_2, \dots, X_n$ from distribution F. An empirical form of $\delta_s$ in Eq. (6) is given by the following:

$$\hat{\delta}_s = \frac{1}{n^2\bar{x}}\sum_i\sum_j\left[\left(\frac{x_i}{1+s} + \frac{1}{(1+s)^2}\right)e^{-x_j(1+s)} - \left(\frac{x_i}{s} + \frac{1}{s^2}\right)e^{-sx_j} + \frac{1+2s}{s^2(1+s)^2}\right]. \quad (10)$$

Let

$$\phi_s(x_1, x_2) = \left(\frac{x_1}{1+s} + \frac{1}{(1+s)^2}\right)e^{-x_2(1+s)} - \left(\frac{x_1}{s} + \frac{1}{s^2}\right)e^{-sx_2} + \frac{1+2s}{s^2(1+s)^2}.$$

Then it is clear to obtain the following

$$\phi_{1,s}(x_1) = E[\phi_s(x_1, x_2)|x_1]$$

$$= \int_0^\infty \phi_1(x_1, x_2)e^{-x_2}dx_2$$

$$= \left(\frac{x_1}{1+s} + \frac{1}{(1+s)^2}\right)\frac{1}{2+s} - \left(\frac{x_1}{s} + \frac{1}{s^2}\right)\frac{1}{1+s} + \frac{1+2s}{s^2(1+s)^2}, \quad (11)$$

and

$$\phi_{2,s}(x_1) = E[\phi_s(x_2, x_1)|x_1]$$

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$$= \int_0^\infty \phi_2(x_2, x_1) e^{-x_2} dx_2$$

$$= e^{-x_1(1+s)} \left(\frac{1}{1+s} + \frac{1}{(1+s)^2}\right) - e^{-sx_1}\left(\frac{1}{s} + \frac{1}{s^2}\right) + \frac{1+2s}{s^2(1+s)^2}. \qquad (12)$$

Hence,

$$\psi_s(x_1) = \phi_{1,s}(x_1) + \phi_{2,s}(x_1)$$

$$= x_1\left(\frac{-2}{s(1+s)(2+s)}\right) + e^{-x_1(1+s)}\left(\frac{2+s}{(1+s)^2}\right) - e^{-sx_1}\left(\frac{1+s}{s^2}\right) + \frac{4s^2+7s+2}{s^2(1+s)^2(2+s)}, \quad s \neq -1, -2, 0. \qquad (13)$$

The expected value and the variance of the test statistic $\hat{\delta}_s$ are as follows:

$E(\psi_s(x_1)) = 0,$ and,

$$\sigma_s^2 = Var(\psi_s(x_1)) = [E(\psi_s(x_1))]^2. \qquad (14)$$

Now $\hat{\delta}_s$ is defined as being equivalent to the formula represented by Lee (1990) as follows:

$$U_n = \frac{1}{\binom{n}{2}} \sum_{i<j} \phi(X_i, X_j). \qquad (15)$$

**Theorem 2.**

**(1)** As n → ∞, the $\sqrt{n}(\hat{\delta}_s - \delta_s)$, is asymptotically normal with mean 0, and variance $\sigma_s^2$ , which is as in Eq. (14).
**(2)** Under $H_0$, the variance of $\hat{\delta}_s$ is

$$\sigma_{0,s}^2 = \frac{7+5s}{(1+s)^2(2+s)^2\big(3+4s(2+s)\big)}. \qquad (16)$$

**Proof:**

(1) and (2) are derived based on the standard theorem of U-statistics (Lee, 1990) and direct calculations, respectively.

**Remark :-**

At special values of s, like as $s = 0.1$ and 0.5, then variance in Eq. (16) is calculated as

$$\sigma_{0,s}^2(0.1) = 0.366, \qquad \sigma_{0,s}^2(0.5) = 0.084.$$

## 2.2. Pitman Asymptotic Efficiency (PAE)

PAE is defined as follows:

$$PAE(\delta_s) = \frac{\left|\frac{d}{d\theta}\delta_\theta\right|_{\theta\to\theta_0}}{\sigma_{0.s}}.$$

Using Eq. (5), we have

$$PAE(\delta_s) = \frac{1}{\sigma_{0.s}}\left|\frac{1}{1+s}\left(\mu_\theta + \frac{1}{1+s}\right)\int_0^\infty e^{-x(1+s)} f_\theta^{\backslash}(x)dx + \int_0^\infty e^{-x(1+s)} f_\theta(x)dx \left(\frac{1}{1+s}\frac{d}{d\theta}\mu_\theta\right)\right.$$
$$\left. - \frac{1}{s}\left(\mu_\theta + \frac{1}{s}\right)\int_0^\infty e^{-sx} f_\theta^{\backslash}(x)dx + \int_0^\infty e^{-sx} f_\theta(x)dx\left(-\frac{1}{s}\frac{d}{d\theta}\mu_\theta\right)\right|_{\theta\to\theta_0} \qquad (17)$$

We compute PAE for the following three alternatives of our class of life distributions.

**(i)** Linear failure rate family:

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$$\bar{F}_1(y) = \exp\left(-y - \frac{\theta y^2}{2}\right), \quad y > 0, \theta \geq 0,$$

**(ii)** Makeham family:

$$\bar{F}_2(y) = \exp\left(-y - \theta(y + e^{-y} - 1)\right), \quad y > 0, \theta \geq 0,$$

**(iii)** Weibull family:

$$\bar{F}_3(y) = \exp\left(-y^{\theta}\right), \qquad y > 0, \theta \geq 0,$$

So, the null hypothesis $H_0$ is obtained at $\theta = 0$ in (i), (ii) and $\theta = 1$ in (iii)

**(i)** PAE($\delta_s$) for linear failure rate family:

$$PAE(\delta_s) = \frac{1}{\sigma_{0.s}}\left|\frac{2s+3}{(2+s)^2(1+s)^2}\right|, \qquad s \neq -2, -1$$

**(ii)** PAE($\delta_s$) for Makeham family:

$$PAE(\delta_s) = \frac{1}{\sigma_{0.s}}\left|\frac{1}{(1+s)(2+s)(3+s)}\right|, \qquad s \neq -3, -2, -1$$

**(iii)** At $s = 0.1, PAE(\delta_s)$ for Weibull family:

$$PAE(\delta_s) = \frac{1}{\sigma_{0.s}}\left|\frac{1}{1+s}\left[\frac{1}{2+s} - 0.628168\right] - \frac{1}{s}\left[\frac{1}{1+s} - 0.611387\right] + [1 - 0.577216]\frac{2}{s(1+s)(2+s)}\right|,$$
$$s \neq 0, -2, -1.$$

Then, $PAE(\delta_s) = \frac{1}{\sigma_{0.s}}(0.54527)$

At $s = 0.5, PAE(\delta_s)$ for Weibull family:

$$PAE(\delta_s) = \frac{1}{\sigma_{0.s}}\left|\frac{1}{1+s}\left[\frac{1}{2+s} - 0.597403\right] - \frac{1}{s}\left[\frac{1}{1+s} - 0.655121\right] + [1 - 0.577216]\frac{2}{s(1+s)(2+s)}\right|,$$
$$s \neq 0, -2, -1.$$

And then, $PAE(\delta_s) = \frac{1}{\sigma_{0.s}}(0.29638)$

In **Table 1**, the Pittman asymptotic efficiencies (PAE) of our test statistics δ^s at s=0.1, 0.5 against previous test statistics are derived.

**Table 1.** PAE for LFR, Makeham and Weibull families

| Test | LFR | Makeham | Weibull |
|---|---|---|---|
| **Bakr et al. (2024)** | 0.815 | 0.153 | ... |
| **Mahmoud and Alim (2008)** | 0.217 | 0.144 | 0.050 |
| **Mahmoud et al. (2019)** | 1.010 | 0.250 | 0.999 |
| **our test $\delta_s(0.1)$** | 1.638 | 0.381 | 1.489 |
| **our test $\delta_s(0.5)$** | 3.386 | 0.907 | 3.527 |

It is clear from the above table that the efficiency of our new class $\delta_s$ of NBULC increases as s increases.

**Table 2** presents the Pittman asymptotic relative efficiency (PARE) of our test $\delta_s$ compared with other test statistics. Note that

$$PARE(T_1, T_2) = \frac{PAE(T_1)}{PAE(T_2)}.$$

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**Table 2.** PARE's of $\delta_s$ with respect to Bakr et al., Mahmoud and Alim and Mahmoud et al.

| Test | | LFR | Makeham | Weibull |
|---|---|---|---|---|
| $\boldsymbol{\delta_s(0.1)}$ | Bakr et al. (2024) | 2.009 | 2.492 | … |
| | Mahmoud and Alim (2008) | 7.550 | 2.649 | 29.796 |
| | Mahmoud et al. (2019) | 1.621 | 1.523 | 1.489 |
| $\boldsymbol{\delta_s(0.5)}$ | Bakr et al. (2024) | 4.153 | 5.932 | … |
| | Mahmoud and Alim (2008) | 15.604 | 6.298 | 70.542 |
| | Mahmoud et al. (2019) | 3.351 | 3.622 | 3.527 |

From **Table 2,** it is shown that the test statistic of our new class (NBULC) $\boldsymbol{\delta_s}$ performs well for Bakr et al. (2024) NBUL, Mahmoud and Alim (2008) NBUFR (new better than used failure rate), and Mahmoud et al. (2019) NBUCL and it is more efficient than (2024), (2008) and (2019) for these families: linear failure rate, Makeham and Weibull.

## 3. POINTS OF CRITICAL VALUE, MONTE CARLO NULL DISTRIBUTION

In **Tables 3** and **4**, we have simulated lower and upper percentile values for 98%, and 99% of $\hat{\delta}_s(0.1)$ and $\hat{\delta}_s(0.5)$ obtained using the Mathematica (12) program. These values are based on 10000 simulated samples of size $n = 5(5)50$.

**Table 3.** The Lower and Upper Percentile of $\hat{\delta}_s(0.1)$

| n | 1% | 5% | 10% | 90% | 95% | 98% | 99% |
|---|---|---|---|---|---|---|---|
| **5** | -0.031432 | -0.114593 | -0.385767 | 0.260868 | 0.321991 | 0.420988 | 0.484287 |
| **10** | -0.087235 | -0.177114 | -0.433382 | 0.188956 | 0.229294 | 0.278298 | 0.309475 |
| **11** | -0.096744 | -0.190446 | -0.434854 | 0.180658 | 0.216796 | 0.260177 | 0.298972 |
| **15** | -0.099859 | -0.181683 | -0.423768 | 0.156097 | 0.187197 | 0.224673 | 0.251661 |
| **16** | -0.105669 | -0.185536 | -0.416104 | 0.154110 | 0.183023 | 0.217597 | 0.242522 |
| **20** | -0.105222 | -0.178390 | -0.376710 | 0.137838 | 0.162334 | 0.196029 | 0.218455 |
| **23** | -0.102886 | -0.172986 | -0.351050 | 0.131584 | 0.156147 | 0.183407 | 0.204895 |
| **25** | -0.103292 | -0.167527 | -0.339916 | 0.126671 | 0.149070 | 0.177677 | 0.195570 |
| **27** | -0.105999 | -0.171716 | -0.329869 | 0.122815 | 0.146327 | 0.171432 | 0.189440 |
| **29** | -0.104227 | -0.168248 | -0.321391 | 0.119388 | 0.141754 | 0.169709 | 0.186451 |
| **30** | -0.102470 | -0.161297 | -0.312873 | 0.118726 | 0.140409 | 0.164797 | 0.182228 |
| **35** | -0.095742 | -0.151484 | -0.302994 | 0.109790 | 0.129887 | 0.151397 | 0.169442 |
| **40** | -0.097113 | -0.149618 | -0.273822 | 0.104276 | 0.121886 | 0.145837 | 0.158182 |
| **43** | -0.091834 | -0.141740 | -0.268037 | 0.102548 | 0.120508 | 0.138775 | 0.154202 |
| **45** | -0.092933 | -0.143269 | -0.262407 | 0.099419 | 0.118084 | 0.139972 | 0.153077 |
| **50** | -0.091379 | -0.139650 | -0.258287 | 0.094922 | 0.112114 | 0.133015 | 0.143435 |

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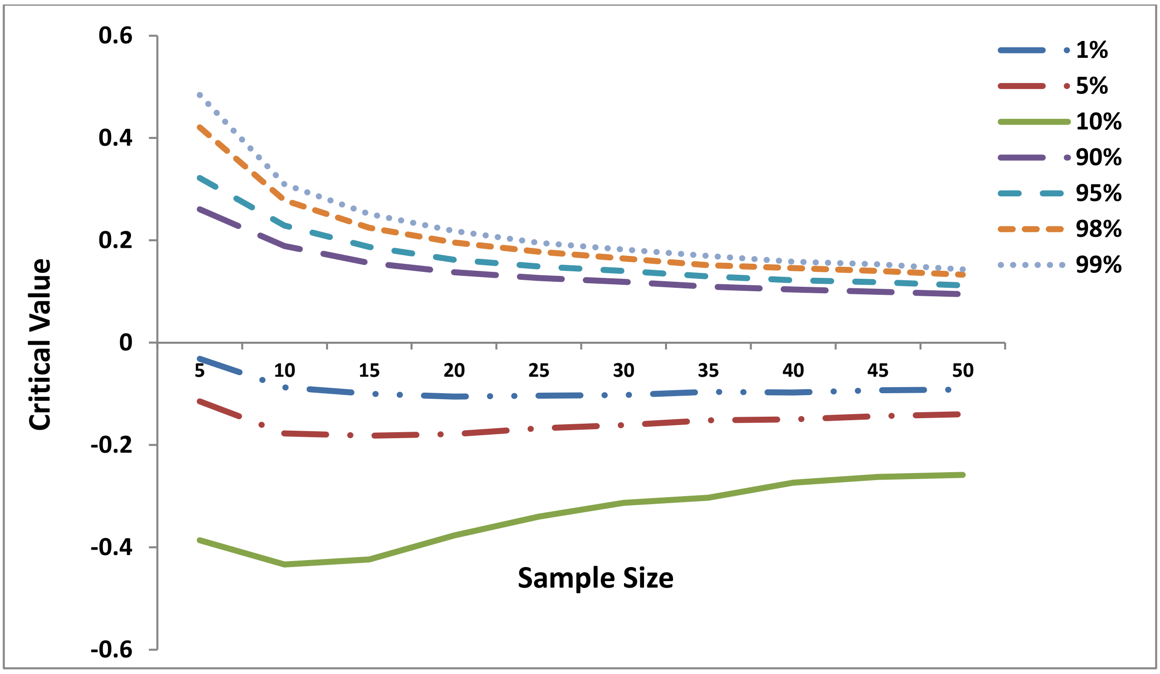


**Figure 1:** The relationship between sample size and critical values of $\hat{\delta}_s$ at s=0.1

As shown in **Table 3** and **Figure 1**, it is clear that the critical values increase gradually with the confidence level, while they decrease as the sample size increases.

**Table 4:** The Lower and Upper Percentile of $\hat{\delta}_s(0.5)$

| n | 1% | 5% | 10% | 90% | 95% | 99% | 99% |
|---|---|---|---|---|---|---|---|
| 5 | -0.043796 | -0.097914 | -0.227378 | 0.144045 | 0.174927 | 0.213068 | 0.239625 |
| 10 | -0.066326 | -0.113996 | -0.230157 | 0.105479 | 0.126185 | 0.150701 | 0.163913 |
| 11 | -0.070533 | -0.113915 | -0.213487 | 0.100887 | 0.12198 | 0.146970 | 0.161291 |
| 15 | -0.063888 | -0.101214 | -0.186705 | 0.089559 | 0.10635 | 0.124370 | 0.138089 |
| 16 | -0.064795 | -0.102068 | -0.183793 | 0.085587 | 0.103341 | 0.122349 | 0.133525 |
| 20 | -0.060273 | -0.093435 | -0.171797 | 0.079013 | 0.094726 | 0.107679 | 0.120223 |
| 23 | -0.061279 | -0.090741 | -0.161192 | 0.073073 | 0.087981 | 0.104441 | 0.116267 |
| 25 | -0.059105 | -0.0885367 | -0.151282 | 0.070449 | 0.083928 | 0.100229 | 0.110816 |
| 27 | -0.059744 | -0.085822 | -0.148729 | 0.0690259 | 0.0826776 | 0.0983598 | 0.104884 |
| 29 | -0.054665 | -0.080537 | -0.130614 | 0.066944 | 0.080022 | 0.092928 | 0.102880 |
| 30 | -0.054136 | -0.081119 | -0.134489 | 0.065578 | 0.078025 | 0.091454 | 0.101156 |
| 35 | -0.052314 | -0.076661 | -0.127114 | 0.061055 | 0.073181 | 0.087271 | 0.096652 |
| 40 | -0.051891 | -0.073724 | -0.122608 | 0.056759 | 0.068342 | 0.080241 | 0.089458 |
| 43 | -0.048067 | -0.068064 | -0.110174 | 0.056746 | 0.067320 | 0.080149 | 0.087127 |
| 45 | -0.047693 | -0.068359 | -0.113313 | 0.054566 | 0.065541 | 0.076858 | 0.086171 |
| 50 | -0.046613 | -0.064645 | -0.106286 | 0.051798 | 0.061972 | 0.073770 | 0.080925 |

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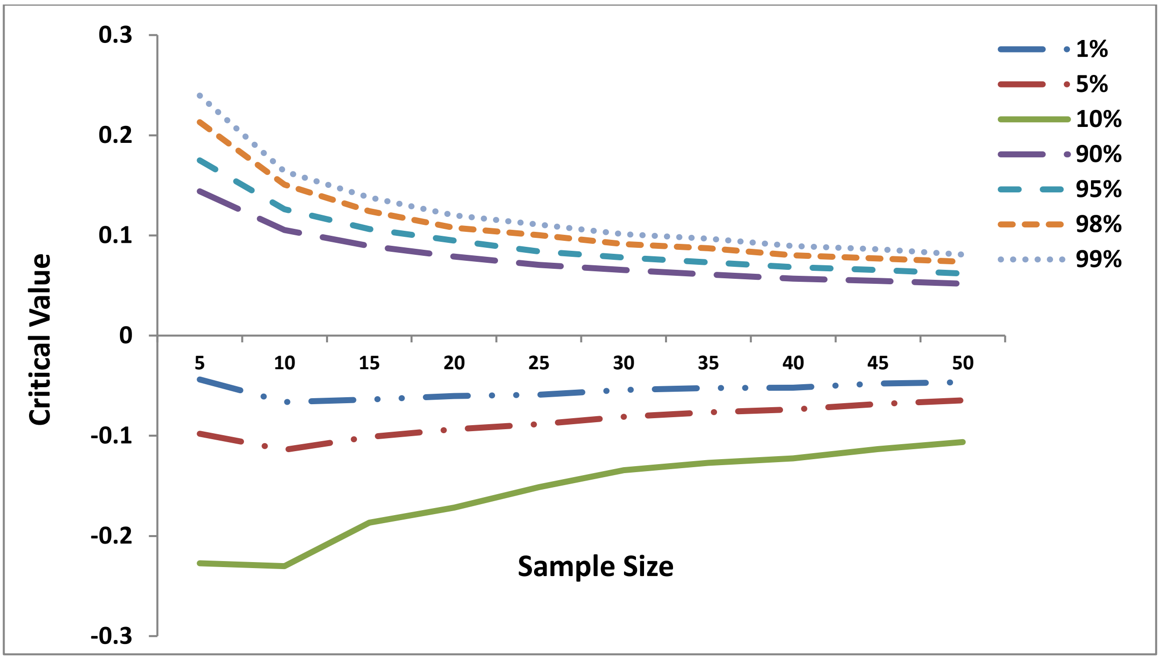


**Figure 2:** The relation between sample size and critical values of $\hat{\delta}_s$ at s=0.5

Also, from the above **Table 4** and **Figure 2**, it is shown that the critical values are increasing as the confidence level increases and decreasing as the sample size increases.

### 3.1. The Power Estimation of the Proposed Test Statistic

The power estimate of our test statistic $\hat{\delta}_s$ is estimated here for significance level α = 0.05, which is based on three of the most widely used alternative distributions, which include:

**(i)** Linear failure rate: $\bar{F}_1(x) = e^{-x-\frac{\theta x^2}{2}}, \qquad x > 0,\ \theta \geq 0$

**(ii)** Makeham: $\bar{F}_2(x) = e^{-x-\theta(x+e^{-x}-1)}, \qquad x > 0,\ \theta \geq 0$

**(iii)** Weibull: $\bar{F}_3(x) = e^{-x^\theta}, \qquad x > 0,\ \theta \geq 0$

**Tables 5** and **6** derive the power estimate with parameter θ = 1,2,3 and 4 at n=10,20 and 30.

**Table 5.** The power estimate of $\hat{\delta}_s(0.1)$

| n | $\theta$ | LFR | Gamma | Weibull |
|---|---|---|---|---|
| | 2 | 1.0000 | 0.9915 | 1.0000 |
| **10** | 3 | 1.0000 | 0.9993 | 1.0000 |
| | 4 | 1.0000 | 1.0000 | 1.0000 |
| | 2 | 1.0000 | 0.9977 | 1.0000 |
| **20** | 3 | 1.0000 | 1.0000 | 1.0000 |
| | 4 | 1.0000 | 1.0000 | 1.0000 |
| | 2 | 1.0000 | 0.9988 | 1.0000 |
| **30** | 3 | 1.0000 | 1.0000 | 1.0000 |
| | 4 | 1.0000 | 1.0000 | 1.0000 |

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**Table 6.** The power estimate of $\hat{\delta}_s(0.5)$

| n | $\theta$ | LFR | Gamma | Weibull |
|---|---|---|---|---|
| **10** | 2 | 1.0000 | 0.9983 | 1.0000 |
| | 3 | 1.0000 | 0.9999 | 1.0000 |
| | 4 | 1.0000 | 1.0000 | 1.0000 |
| **20** | 2 | 1.0000 | 0.9997 | 1.0000 |
| | 3 | 1.0000 | 1.0000 | 1.0000 |
| | 4 | 1.0000 | 1.0000 | 1.0000 |
| **30** | 2 | 1.0000 | 1.0000 | 1.0000 |
| | 3 | 1.0000 | 1.0000 | 1.0000 |
| | 4 | 1.0000 | 1.0000 | 1.0000 |

It is shown from the above **Tables 5** and **6** the power estimates of the test statistic increases when the parameter $\theta$ and the sample size n increase, so we note that our test plays a good power.

## 4. TESTING FOR CENSORED DATA

The objective of this section is to conduct a test, in the case of randomly right-censoring data, comparing two hypotheses. Specifically, the null hypothesis $H_0$ assumes that survival distribution follows a constant failure rate. In contrast, the alternative hypothesis $H_1$: the survival distribution follows the NBULC model of life distribution.

The observations available in the life model of testing or clinical study are the censored data, where the observations may be lost or censored before completing the study. The experiment is described as the following: Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (i.i.d) to the continuous life distribution F, and $Y_1, Y_2, \dots, Y_n$ be independent identically distributed (i.i.d) to the continuous life distribution G. Suppose that the Xs and Ys are independent. In a randomly right-censored model, we note that the pairs $(Z_j,\delta_j)$, j=1,2,...,n, where $Z_j = min(X_j, Y_j)$ and

$$\delta j = \begin{cases} 1, \text{if } \ Zj \ = \ Xj \text{ (jth observed is uncensored)} \\ 0, \text{if } \ Zj \ = \ Yj \ \text{ (jth observed is censored)} \end{cases}$$

Let $Z_{(0)} = 0 < Z_{(1)} < Z_{(2)} < \dots < Z_{(n)}$ denote the order Z's and $\delta_{(j)}$ is $\delta_j$ corresponding to $Z_{(j)}$. Kaplan(1958) Proposed a product limit estimator based on the censored data $Z_j,\delta_j$, j= 1,2,...,n, as the following

$$\bar{F}_n(X) = \prod_{[j;Z_j \le X]} (n-j)/(n-j+1)^{\delta(j)}, X \in [0, Z_{(n)}].$$

Using Eq. (6) we obtain the following test statistic

$$\hat{\delta}_s^c = \frac{1}{1+s}\left[\varphi + \frac{1}{1+s}\right]\vartheta - \frac{1}{s}\left[\varphi + \frac{1}{s}\right]\Omega, \qquad (18)$$

where,

$$\varphi = \sum_{k=1}^{n}\prod_{m=1}^{k-1} C_m^{\delta(m)} \left[Z_k - Z_{k-1}\right],$$

$$\vartheta = \sum_{j=1}^{n} e^{-(1+s)Z_j}\left[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}\right],$$

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$$\Omega = \sum_{i=1}^{n} e^{-sZ_i}\left[\prod_{v=1}^{i-2} C_v^{\delta(v)} - \prod_{v=1}^{i-1} C_v^{\delta(v)}\right],$$

and,

$$dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_j), \qquad C_k = \frac{n-k}{n-k+1}$$

Now, we perform a Monte Carlo simulation to estimate the critical points of the null distribution for the test statistic $\delta^{\hat{}}$ as defined in Eq. (18). The simulation is carried out by generating 10000 data set from a standard exponential distribution. Specifically, the simulated datasets are based on the following samples: 10(5), 30 (10), 81, and 86.

By using the Mathematica program (12). **Tables 7** and **8** give the upper and the lower percentile points for 1%, 5%, 10%, 90%, 95%, 99% of the statistic $\hat{\delta}_s^c$.

**Table 7.** The critical values of $\hat{\delta}_s^c(0.1)$ with 10000 replications.

| n | 1% | 5% | 10% | 90% | 95% | 98% | 99% |
|---|---|---|---|---|---|---|---|
| **10** | -0.062349 | -0.031897 | 0.054316 | 0.493805 | 0.583887 | 0.669347 | 0.739960 |
| **15** | -0.031341 | -0.023025 | 0.007383 | 0.186262 | 0.218850 | 0.259220 | 0.284908 |
| **20** | -0.018766 | -0.015897 | -0.008059 | 0.092754 | 0.108723 | 0.127443 | 0.138493 |
| **25** | -0.012372 | -0.010794 | -0.007709 | 0.052355 | 0.061748 | 0.072415 | 0.079583 |
| **30** | -0.008752 | -0.007894 | -0.006498 | 0.033257 | 0.038932 | 0.046407 | 0.051554 |
| **40** | -0.004999 | -0.004580 | -0.003809 | 0.016338 | 0.019167 | 0.022749 | 0.025487 |
| **50** | -0.003247 | -0.003024 | -0.002665 | 0.009138 | 0.010777 | 0.012719 | 0.013833 |
| **60** | -0.002258 | -0.002140 | -0.001939 | 0.005867 | 0.006947 | 0.008093 | 0.008969 |
| **70** | -0.001677 | -0.001595 | -0.001477 | 0.003954 | 0.004728 | 0.005639 | 0.006208 |
| **81** | -0.001257 | -0.001195 | -0.001109 | 0.002735 | 0.003243 | 0.003814 | 0.004204 |
| **86** | -0.001121 | -0.001069 | -0.000999 | 0.002346 | 0.002789 | 0.003303 | 0.003695 |

From **Table 7**, it is evident that the critical points for the test statistic exhibit a gradual decrease as the sample sizes increase. This observation suggests that larger sample sizes lead to more stable estimates of the test statistic, resulting in lower critical values at the same significance levels. And this is shown in the following figure.

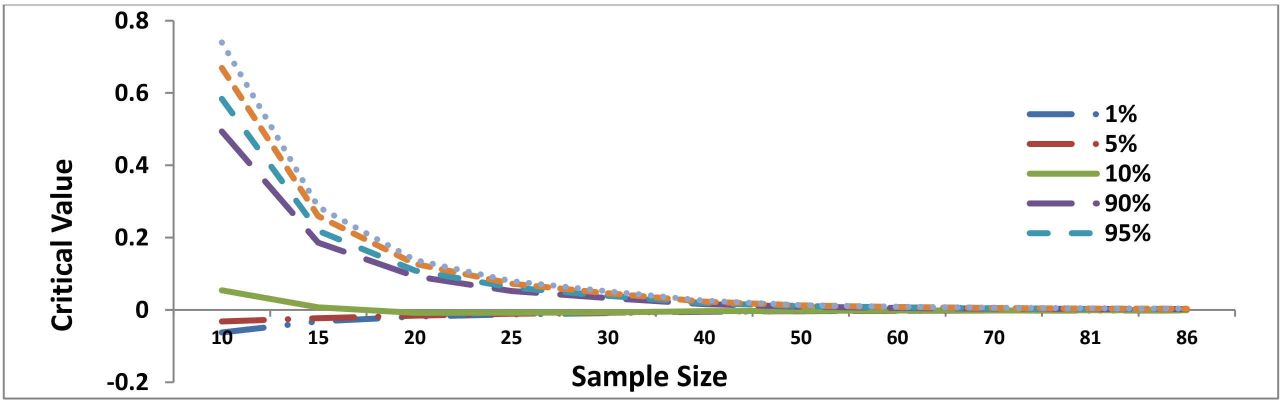


**Figure 3:** The Relation between Sample Size and Critical Values of $\hat{\delta}_s^c$ at s=0.1

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**Figure 3** offers a clear visual representation of two fundamental trends in hypothesis testing:

1. Critical Values Increase as the Confidence Level Increases.
2. Critical Values Decrease as Sample Size Increases.

Also, this is shown in the following table and figure.

**Table 8.** The critical values of $\hat{\delta}_s^c(0.5)$ with 10000 replications.

| n | 1% | 5% | 10% | 90% | 95% | 98% | 99% |
|---|---|---|---|---|---|---|---|
| **10** | -0.008207 | -0.005446 | -0.002053 | 0.015535 | 0.018868 | 0.022837 | 0.025602 |
| **15** | -0.004156 | -0.003374 | -0.001955 | 0.005824 | 0.006935 | 0.008295 | 0.009238 |
| **20** | -0.002425 | -0.002069 | -0.001495 | 0.002789 | 0.003356 | 0.004058 | 0.004508 |
| **25** | -0.001603 | -0.001400 | -0.001109 | 0.001612 | 0.001936 | 0.002364 | 0.002647 |
| **30** | -0.001131 | -0.001013 | -0.000860 | 0.000982 | 0.001185 | 0.001402 | 0.001544 |
| **40** | -0.000646 | -0.000589 | -0.000513 | 0.000473 | 0.000568 | 0.000666 | 0.000734 |
| **50** | -0.000417 | -0.000386 | -0.000344 | 0.000264 | 0.000323 | 0.000389 | 0.000431 |
| **60** | -0.000292 | -0.000271 | -0.000246 | 0.000165 | 0.000199 | 0.000240 | 0.000268 |
| **70** | -0.000216 | -0.000202 | -0.000187 | 0.000109 | 0.000131 | 0.000159 | 0.000176 |
| **81** | -0.000161 | -0.000153 | -0.000143 | 0.000074 | 0.000089 | 0.000109 | 0.000123 |
| **86** | -0.000144 | -0.000135 | -0.000127 | 0.000064 | 0.000077 | 0.000094 | 0.000105 |

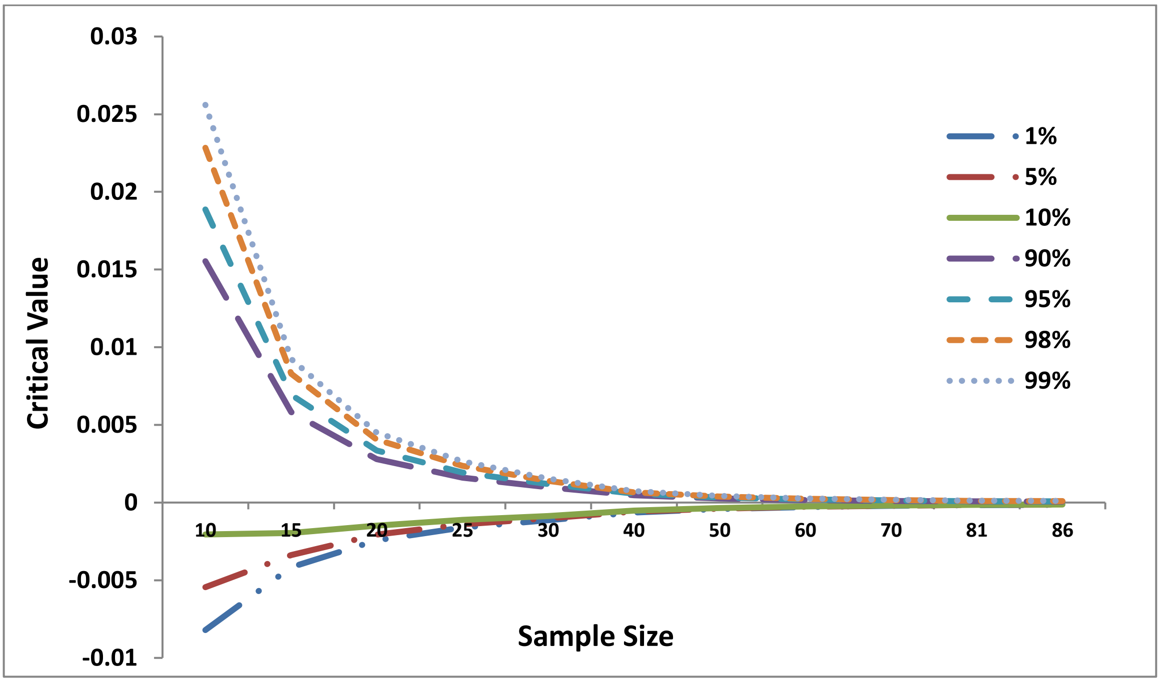


**Figure 4:** The Relation between Sample Size and Critical Values of $\hat{\delta}_s^c$ at s=0.5

Above the table and figure lead to the critical points decreasing slowly as the sample sizes increase, and also they increase as the confidence levels increase, and it is shown in the following figure.

### 4.1. Power of a Test

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The power of a hypothesis test is defined as the probability that the test will correctly reject the null hypothesis when it is false.

At significance level $\alpha$ = 0.05, the power of the proposed test $\hat{\delta}_s^c(0.5)$ is calculated with respect to three alternative distributions (linear failure rate (LFR), gamma, and Weibull ). Using the Mathematica (12) program, that is based on 10000 samples.

**Table 9** presents the power estimates for various parameter values $\theta$ = 1,2,3 and 4 at n=10,20 and 30.

**Table 9.** The power estimate of $\hat{\delta}_s^c(0.5)$ of censored.

| n | $\theta$ | LFR | Gamma | Weibull |
|---|---|---|---|---|
| | 2 | 0.9997 | 0.9484 | 0.9999 |
| 10 | 3 | 0.9995 | 0.9888 | 1.0000 |
| | 4 | 0.9996 | 0.9997 | 1.0000 |
| | 2 | 0.9999 | 0.9964 | 0.9997 |
| 20 | 3 | 1.0000 | 1.0000 | 1.0000 |
| | 4 | 1.0000 | 1.0000 | 1.0000 |
| | 2 | 0.9999 | 0.9999 | 1.0000 |
| 30 | 3 | 1.0000 | 1.0000 | 1.0000 |
| | 4 | 1.0000 | 1.0000 | 1.0000 |

The results in **Table 9** demonstrate that the estimated power of our test increases as both the sample size n and the parameter θ increase.

## 5. SCIENTIFIC DATA APPLICATIONS

We give several real-medical examples to illustrate the applications of our test at the 95% confidence level.

### 5.1. Case of complete data

**Application 1.**

Consider the data presented in Abouammoh (1994), which consists of a set of 40 patients diagnosed with blood cancer (leukemia) at a Ministry of Health hospital in Saudi Arabia. The following are the ordered values of their survival times, measured in years:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.315 | 0.496 | 0.616 | 1.145 | 1.208 | 1.263 | 1.414 | 2.025 | 2.036 | 2.162 |
| 2.211 | 2.370 | 2.532 | 2.693 | 2.805 | 2.910 | 2.912 | 3.192 | 3.263 | 3.348 |
| 3.348 | 3.427 | 3.499 | 3.534 | 3.767 | 3.751 | 3.858 | 3.986 | 4.049 | 4.244 |
| 4.323 | 4.381 | 4.392 | 4.397 | 4.647 | 4.753 | 4.929 | 4.973 | 5.074 | 4.381 |

We get, at **s=0.1,** $\hat{\delta}_s$**= 0.845732** and at **s=0.5,** $\hat{\delta}_s$**= 0.320997,** which are greater than the critical values in **Tables 3** and **4** in both cases, we reject the null hypothesis, which asserts that the dataset follows the NBULC distribution and not the exponential distribution.

**Application 2.**

Using the dataset provided in Grubbs (1971), the data represents the time intervals between the arrivals of 25 customers at a facility.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.80 | 2.89 | 2.93 | 3.03 | 3.15 | 3.43 | 3.48 | 3.57 | 3.85 | 3.92 |
| 3.98 | 4.06 | 4.11 | 4.13 | 4.16 | 4.23 | 4.34 | 4.73 | 4.53 | 4.62 |
| 4.65 | 4.84 | 4.91 | 4.99 | 5.17 | | | | | |

We get that $\hat{\delta}_s$ = **1.27018**, at **s=0.1,** $\hat{\delta}_s$ = **0.451456**, at **s=0.5,** which are exceed the critical value in **Tables 3** and **4** in two cases. Then we reject the null hypothesis, which asserts that the dataset follows the NBULC distribution rather than the exponential distribution.

**Application 3.**

Consider the following data, which represent the failure times in hours for a particular type of electrical insulation in an experiment where the insulation was exposed to a continuously increasing voltage stress see Lawless (1982).

0.205 0.363 0.407 0.770 0.720 0.782 1.178 1.255 1.592 1.635 2.310

It is easy to show that $\hat{\delta}_s$=**0.165757,** at **s=0.1,** $\hat{\delta}_s$ = **0.106036,** at **s=0.5,** which are less the critical value in **Tables 3** and **4** in two cases. Then we accept $H_0$ , which asserts that the dataset follows an exponential distribution.

### 5.2. Case of Censored Data

**Application 4.**

Consider the dataset from Mahmoud et al. (2005), which includes 51 liver cancer patients admitted to the El Minia Cancer Center, Ministry of Health Egypt, in 1999. The following are the ordered lifetimes (in days) for the non-censored observations.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10 | 14 | 14 | 14 | 14 | 14 | 15 | 17 |
| 18 | 20 | 20 | 20 | 20 | 20 | 23 | 23 |
| 24 | 26 | 30 | 30 | 31 | 40 | 49 | 51 |
| 52 | 60 | 61 | 67 | 71 | 74 | 75 | 87 |
| 96 | 105 | 107 | 107 | 107 | 116 | 150 | |

The ordered censored data:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 30 | 30 | 30 | 30 | 30 | 60 | 150 | 150 |
| 150 | 150 | 150 | 185 | | | | |

In this case $\hat{\delta}^c_s$=**-0.000230831,** at **s=0.1,** $\hat{\delta}^c_s$= **−8.58783∗10⁻⁸**, at **s=0.5,** which are less than the critical value of **Tables 7** and **8.** Hence, the null hypothesis asserting that the data set does not exhibit the NBULC (New Better than Used in the Laplace convex order) property is rejected.

## 6. CONCLUSION

A new class of life distributions, known as NBULC (New Better than Used in the Laplace-Convex), is introduced. Using a goodness offit framework, novel test statistic is developed to test exponentiality against the NBULC class for both non-censored and censored data. Our class NBULC of life distribution is the largest, and the proposed test statistic is found to be more efficient than other tests and has good power for other alternative classes of life distributions; our test's critical values are given. Finally, real data sets are applied to show the usefulness of our test statistics.

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