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Research Article

A Study on Mode Graph Labeling

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ABSTRACT

Mode graph labeling is an essential concept in graph theory, focusing on labeling the vertices or edges of a graph to determine the most frequent label under specific constraints. This paper provides a comprehensive overview of mode graph labeling, and introduces key definitions and theorems. Theoretical frameworks and significant theorems are examined to provide a deeper understanding of their applications in graph theory. These insights pave the way for advancements in combinatorial optimization and network design.

Keywords: Mode graph labeling, Mode vertex labeling, Mode edge labeling, Mode total labeling.

1. INTRODUCTION

Assigning labels to a graph's edges, vertices, or both—typically represented by integers—is known as graph labeling in mathematical field of graph theory [1]. Vertex labeling in a graph G=(V, E) is a function of V to a collection of labels. A vertex-labeled graph is one that has such a function defined [2]. A function of E to collection of labels is called edge labeling. The graph can be referred to as an edge-labeled graph in this instance.

Considering G to be graph with a labeling function $f : X \to Z^+$, where X is either V(vertices), E(edges), or both. The mode of graph G, denoted as Mode(G), is label that occurs with the highest frequency. Mode graph labeling finds applications in fields such as network optimization, resource allocation, coding theory, and data analysis, making it a cornerstone of both theoretical and applied graph research

2. MODE GRAPH LABELING

2.1 Definition : Labeling function

A labeling function f assigns a positive integer to each element $x \in X$, where X⊆V∪E $f : X \to Z^+,\ f(x) = k, k \in Z^+$

2.2 Definition: Label Frequency Function

The label frequency function $\phi: Z^+ \to N$ Computes the number of elements assigned a specific label k that contains $\phi(k) = |\{x \in X: f(x) = k\}|$.

Mode graph labeling can be classified into three

3. MODE VERTEX LABELING

Considering $G = (V, E)$ “be the graph where V represent the set of vertices and E represent set of edges” f_v be the vertex labeling function $f_v: V \to Z^+$assigns a positive integer to each vertex $v \in V$, $f_v(v) = k, k \in Z^+$, then define the label frequency function for each label k is $\phi: Z^+ \to N$ such that $\phi(k) = |\{ v \in V: f_v(v) = k \}|$ represent the number of vertices labeled with k then vertex mode of G, $\text{Mode}_V(G) = \arg max_{k\in Z^+} \phi(k)$

3.1 Example : Examine the graph G = (V,E) with vertex labeling function f_V assigns labels as follows

$$f_V(v_1)= 2,\ f_V(v_2)=3,\ f_V(v_3)= 2,\ f_V(v_4)= 1$$

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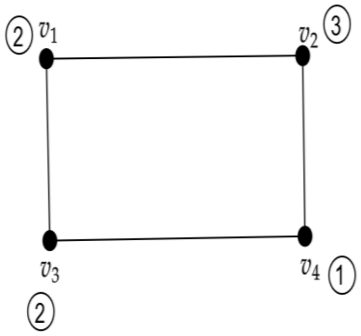

Figure 3.1 Mode vertex labeling

From figure 3.1 the frequency map ϕ would be ϕ (1) = 1, ϕ (2) = 2, ϕ (3) = 1

Hence vertex mode of the graph G is 2.

3.2 Theorem: Uniqueness of Vertex Mode

Regarding $G = (V, E)$ if no two labels have the same maximum frequency, then the vertex mode , $Mode_V$ (G) is unique.

Proof

Assume that $Mode_V$ (G) is not unique this implies there exist two distinct labels k_1 and k_2 that contains $\phi(k_1)$ = $\phi(k_2)$ = arg $max_{k\in Z^+}\phi(k)$ then by the definition of the mode

$Mode_V$ (G) = { k_1, k_2}

This contradicts the assumption that no two labels have the same maximum frequency.
Since $\phi(k_1) = \phi(k_2)$ but $k_1 \neq k_2$, this violates the condition $\phi(k_1) \neq \phi(k_2)$ for distinct k_1 and k_2. Since no two labels can share the same maximum frequency, there exists exactly one label k such that $\phi(k)$ = arg $max_{j\in Z^+}\phi(j)$, Thus $Mode_V$ (G) = k is unique.

4. MODE EDGE LABELING

Considering $G = (V, E)$ be a graph where V represent the set of vertices and E represent the set of edges. f_E be the edge labeling function $f_E: E \to Z^+$, assigns a positive integer to each edge $e \in E$, $f_E\,(e) = k, k \in Z^+$then define the label frequency function $\phi: Z^+ \to N$ such that ϕ (k) = | { e € E: $f_E\,(e) = k$ }| represent number of edges labeled with k then edge mode of G , $Mode_E$ (G) = arg $max_{k\in Z^+}\phi(k)$

4.1 Example : Consider about the graph G = (V,E) with edge labeling function f_E assigns labels as follows

$$f_E(e_1)= 5 , f_E(e_2)= 5, f_E(e_3)= 2, f_E(e_4)= 3, f_E(e_5)= 5$$

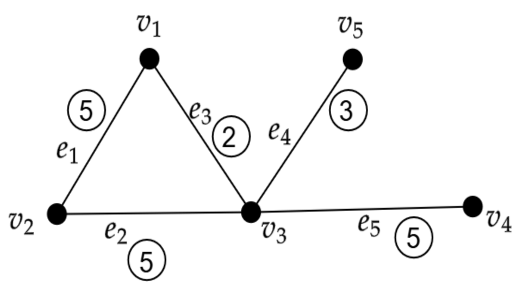

Figure 3.2 : Mode edge labeling

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From figure 3.2 the frequency map ϕ would be *ϕ (5) = 3, ϕ (2) = 1, ϕ (3) = 1*,Hence the edge mode of the graph G is 5.

4.2 Theorem: Uniqueness of Edge Mode

Regarding "a graph $G = (V, E)$ with an edge labeling function $f_E: E \to Z^+$, if" no two labels have the same maximum frequency, then the edge mode $Mode_E$ (G) is unique.

Proof

Assume that $Mode_E$ (G) is not unique this implies there exist two distinct labels k_1 & k_2 that contains $\phi(k_1) = \phi(k_2) = \arg max_{k\in Z^+}\phi(k)$ then by the definition of the mode

$Mode_E$ (G) = { k_1, k_2}

This assumption directly contradicts the hypothesis that no two labels can have the same maximum frequency.If $\phi(k_1) = \phi(k_2)$ but $k_1 \neq k_2$, this violates condition $\phi(k_1) \neq \phi(k_2)$ for distinct k1 and k2. Since no two labels can share the same maximum frequency, there exists exactly one label k such that ϕ(k) = $\arg max_{j\in Z^+}\phi(j)$, Thus $Mode_E$ (G) = k is unique.

5. MODE TOTAL LABELING

Considering "G = (V, E) be graph with vertex Set V and edge set E define the" following.

5.1 Definition: Vertex labeling function $f_v : V \to Z^+$ assigns a positive integer label to each vertex $v \in V$

5.2 Definition: Edge labeling function $f_E : E \to Z^+$ assigns a positive integer label to each edge $e \in E$

5.3 Definition: Combine label function

$$C : V \cup E \to Z^+ \text{Such that } ((x) = \begin{cases} f_v(x) & y\, x \in V \\ f_e(x) & y\, x \in E \end{cases}$$

5.4 Definition: combined label frequency function $\phi : Z^+ \to N$ defined as

$$\phi(k) = \left|\{x \in V \cup E / (x \in V \text{ and } f_v(x) = k) \text{ or } x \in E \text{ and } f_e(x) = k\}\right|$$

Which counts the no: of vertices as edges labeled with "k.

Considering $G = (V, E)$ be graph with vertex set V and edge set E consider the vertex labeling" function be $f_v: V \to Z^+$and the edge labeling function be $f_E: E \to Z^+$, define the combine label *C: VUE* → *Z*⁺ then define the combine label frequency function ϕ: $Z^+ \to N$ that contains ϕ (k) = |{x ∈ V∪E , (x∈V and f_v(x)=k) or (x ∈E and f_E (x)=k)}| is number of vertices or edges labeled with k then total mode of G , Mode (G) = $\arg max_{z\in Z^+}\phi(k)$

5.5 Example :

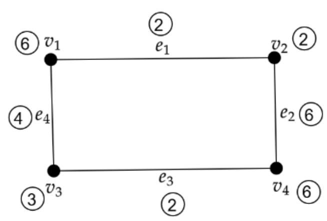

Figure 3.3 : Mode total labeling

In figure 3.3 Vertex labels are $f_V(v_1)$= 6, $f_V(v_2)$=2, $f_V(v_3)$= 3, $f_V(v_4)$= 6

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Edge labels are $f_E(e_1)= 2, f_E(e_2)= 6, f_E(e_3)= 2, f_E(e_4)= 4$

Combine labels are $\{ f_V(v_1)= 6,\ f_V(v_2)=2, f_V(v_3)= 3, f_V(v_4)= 6, f_E(e_1)= 2, f_E(e_2)= 6, f_E(e_3)= 2, f_E(e_4)= 4 \}$

Now combine label frequency function $\phi(6) = 2, \phi(2) = 1, \phi(3) = 1, \phi(4) = 1, \phi(2) = 2,$

$\phi(6) = 1$

Mode (G) = $\{6,2\}$

5.6 Theorem

If k^* is the total mode of G, then the combined label frequency at k^*, $\phi(k^*)$ satisfies $\phi(k^*) = \phi_v(k^*) + \phi_e(k^*)$

Proof

The total mode (G) of a graph G = (V, E) with labeling function $\phi_v : V \to Z^+$ and edge labeling function $\phi_e : E \to Z^+$

By the definition of $\phi(k)$

We've $\phi(k) = \left|\{x \in V \cup E / (x \in V \text{ and } f_v(x) = k) \text{ or } (x \in E \text{ and } f_e(x) = k)\}\right|$

The Set $\{x \in V \cup E\}$ can be partitioned into $\{x \in V\}$ and $\{x \in E\}$ So

$$\phi(k) = \underbrace{\left|\{v \in V / f_v(v) = k\}\right|}_{\phi_v(k)} + \underbrace{\left|\{e \in E / f_e(e) = k\}\right|}_{\phi_e(k)}$$

For "k = k^* = Mode (G), It follows directly that

$$\phi(k^*) = \phi_v(k^*) + \phi_e(k^*)$$

where $\phi_v(k^*) = \left|\{v \in V / f_v(v) = k^*\}\right|$ is" the frequency of k^* among the vertex labels and

$\phi_e(k^*) = \left|\{e \in E / f_e(e) = k^*\}\right|$ is the frequency of k^* among the edge labels.

5.7 Conclusion

Mode graph labeling represents a significant concept in graph theory, focusing on the efficient assignment of labels to vertices and edges under specific constraints to analyze the frequency distribution of labels. The main contributions of this study include definitions and foundational Concepts, classification of mode graph labeling, and theoretical Implications. By examining definitions and proving critical theorems, this work provides a pathway for advanced research in combinatorial structures and optimization.

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