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An enhanced rectangular Membrane Finite Element Based on the Strain Approach

Bouanaka Mohamed Larbi[1], Rebiai Cherif[2] ,Saidani Nourreddine[3]
[123] *Laboratory of Innovation in Construction, Eco-design, and Seismic Engineering, Department of Mechanical Engineering, Faculty of Technology, University of Batna2-Mostefa Ben Boulaid, Batna 05000, Algeria.*
**Corresponding Author: Mohamed.bouanaka@univ-batna2.dz*
c.rebiai@univ-batna2.dz
n.saidani@univ-batna2.dz

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ABSTRACT

This study presents the development of a enhanced rectangular membrane finite element using the strain-based approach, designed for both elastic and elastoplastic analyses. Each of the four corner nodes of the element is characterized by three degrees of freedom: two translational degrees and one in-plane rotational degree. The displacement functions of the proposed element accurately capture the rigid body motion modes. For the elastoplastic computations, the material response is evaluated using the Von Mises failure creterion, and the nonlinear solution procedure incorporates viscoplastic-strain approaches. For validation purposes, several selected numerical examples are solved using the developed element in both linear and nonlinear analyses.

The results obtained are compared with analytical solutions and other similar membrane finite elements, demonstrating the strong performance of the proposed element.

Keywords: strain approach, membrane element, nonlinear analysis, drilling rotation, linear analysis.

1 INTRODUCTION

The strain-based formulation has been widely employed by researchers in the development of new finite elements, primarily aiming to enhance their accuracy and efficiency.Initially, this method was applied to elements with curved geometries [1, 2]. Over time, it was extended to cover applications in plane elasticity [3–5], three-dimensional elasticity [6], cylindrical shells [7–10], and plate bending problems [11, 12]. This approach is especially valuable for addressing both linear and nonlinear analyses within the finite element framework. Given that exact analytical solutions exist only for a limited range of simple problems, finite element methods become indispensable.

Despite their widespread use, traditional displacement-based elements are proving to be increasingly inefficient, often resulting in prolonged computation times. In contrast, strain-based elements have shown superior performance in various studies [13-17], outperforming displacement-based elements in several aspects. In light of these advantages, an enhanced strain-based membrane element incorporating drilling degrees of freedom—named “SBME” (**S**train **B**ased **M**embrane **E**lement) is proposed. This element is suitable for both static and elastoplastic analyses of membrane structures. To assess its effectiveness, a series of benchmark numerical examples were analyzed. The results demonstrate that the proposed element yields results that align well with both analytical and previously published numerical solutions.

2. Theoretical considerations

The figure 1 schematically illustrates the SBME element, which features three degrees of freedom at each of its four corner nodes: two translational components (U_i, V_i) and one in-plane rotational component (θ_i).

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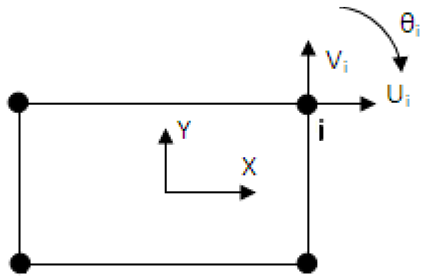

Figure 1. SBME element with drilling rotation

Equation (1) provides the three strain components for a two-dimensional linear analysis.

$$\varepsilon_x = \frac{\partial U}{\partial x} \qquad (1)$$
$$\varepsilon_y = \frac{\partial V}{\partial y}$$
$$\gamma_{xy} == \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}$$

The strain components presented in Equation (1) are not independent, as they are expressed in terms of the two displacement variables, U and V. Therefore, the strain must satisfy an additional condition known as the compatibility equation. This equation is derived by eliminating U and V from Equation (1), resulting in:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \qquad (2)$$

We first integrate Eq. (1) with all three strains equal to zero to obtain:

$$U = a_1 - a_3 y$$
$$V = a_2 + a_3 x \qquad (3)$$
$$\theta = a_3$$

As indicated in Equation (3), the parameters (a_1, a_2, a_3), account for the rigid-body motion modes. After separating these non-deformational contributions, nine independent coefficients (a_4, a_5,.....,a_{12}) remain available to characterize the deformation-induced kinematics of the element. These coefficients are distributed among the strain components according to the adopted interpolation scheme. Consequently, the corresponding strain and displacement approximations can be expressed as follows:

$$\varepsilon_x = a_4 + 2a_5 x + a_6 y + a_{10} y^2 + 2a_{11} x y^3$$
$$\varepsilon_y = a_7 + a_8 x - a_{10} x^2 - 2a_{11} y x^3 \qquad (4)$$
$$\gamma_{xy} = 2a_5 + 2a_6 x + 2a_7 x + 2a_8 y + 2a_9 y + 2a_{12} x$$

The compatibility condition of Equation (2) is inherently satisfied by the strain field presented in Equation (4). Substituting the strain–displacement relationships from Equation (1) and integrating the resulting system yields the following expressions.

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$$
\begin{aligned}
U &= a_4 x + a_5(x^2+y) + a_6 xy + \\
&\quad a_8 y^2/2 + a_9 y^2 + a_{10}(xy^2) + a_{11}x^2y^3 \\
V &= a_5 x + a_6 x^2/2 + a_7(x^2+y) + a_8 xy - \\
&\quad a_{10}(x^2y) - a_{11}x^3y^2 + a_{12}x^2 \\
\theta &= -a_9 y - 2a_{10}(xy) - 3a_{11}x^2y^2 + a_{12}x
\end{aligned}
\tag{5}
$$

The final displacement functions are obtained by adding Eq. (3) and (5) to obtain the following:

$$
\begin{aligned}
U &= a_1 - a_3 y + a_4 x + a_5(x^2+y) + a_6 xy + \\
&\quad a_8 y^2/2 + a_9 y^2 + a_{10}(xy^2) + a_{11}x^2y^3 \\
V &= a_2 + a_3 x + a_5 x + a_6 x^2/2 + a_7(x^2+y) + a_8 xy - \\
&\quad a_{10}(x^2y) - a_{11}x^3y^2 + a_{12}x^2 \\
\theta &= a_3 - a_9 y - 2a_{10}(xy) - 3a_{11}x^2y^2 + a_{12}x
\end{aligned}
\tag{6}
$$

The stiffness matrix [Ke] for the present element is given by

$$\left[K^e\right] = \left[C\right]^{-T}\left(\int_{-1}^{1}\int_{-1}^{1}[Q]^T[D][Q]\det|J|\,d\xi\,d\eta\right)[C]^{-1} \tag{7}$$

Where [Q], [J] and [D] are the strain, the elasticity and the Jacobian matrices respectively, and [C] is the matrix which relates the 12 nodal displacements to the 12 constants a_1 to a_{12}. These are given respectively by

$$[Q] = \begin{bmatrix} 0 & 0 & 0 & 1 & 2x & y & 0 & 0 & 0 & y^2 & 2xy^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & 0 & -x^2 & -2x^3y & 0 \\ 0 & 0 & 0 & 0 & 2 & 2x & 2x & 2y & 2y & 0 & 0 & 2x \end{bmatrix} \tag{8}$$

$$[D] = \frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \tag{9}$$

For the case of plane stress problems the elasticity matrix [D] is:

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} (1-v) & v & 0 \\ v & (1-v) & 0 \\ 0 & 0 & \frac{(1-2v)}{2} \end{bmatrix} \tag{10}$$

$$[C] = \begin{bmatrix} 1 & 0 & -y & x & x^2+y & xy & 0 & 0.5y^2 & y^2 & xy^2 & x^2y^3 & 0 \\ 0 & 1 & x & 0 & x & x^2/2 & x^2+y & xy & 0 & -(x^2y) & -x^3y^2 & x^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -y & -2xy & -3x^2y^2 & x \end{bmatrix} \tag{11}$$

3. LINEAR NUMERICAL VALIDATION

3.1.Mac-Neal's elongated cantilever beam

The cantilever structure illustrated in Fig. 2 corresponds to a benchmark example originally investigated by Mac-Neal and Harder [18]. In this study, the free extremity of the beam is loaded by a unit transverse force (P=1) together

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with an applied end moment of magnitude M=10. The material is assumed to be linear elastic, characterized by a Young's modulus of 10^7 and a Poisson's ratio equal to 0.3, while the beam thickness is taken as t=0.1.

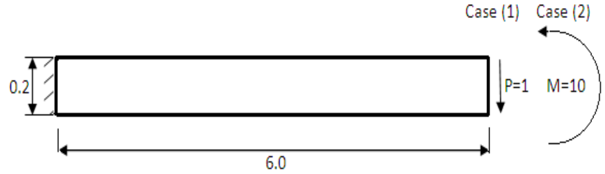

Figure 2. Mac-Neal's elongated beam subject to (1) end shear and (2) end bending

Examination of the normalized free-end deflections listed in Tables 1 and 2 demonstrates the effectiveness of the SBME element. Compared with SBRIEIR [19], SBRIE [20], Q8, and T6, the proposed formulation yields results that remain closer to the reference values. In addition, under both the tip shear force and the pure bending moment, the numerical predictions obtained with SBME converge consistently toward the exact solution. Another advantage is its reduced number of degrees of freedom relative to Q8 and T6, which decreases the computational effort while maintaining a high level of accuracy.

TABLE I . Normalised deflection for MAC-NEAL'S elongated beam subjected to end shear

Load case (1): Force shearing at the free end P=1					
Mesh	T6	Q8	SBRIE	SBRIEIR	**SBME**
6 x 1	0.983	0.9868	0.9035	0.9035	**0.96**
12x1	0.993	0.9933	0.9083	0.9083	**0.989**
Reference solution[18]			1,000 (0.1081)		

TABLE III. Normalised deflection for MAC-NEAL'S elongated beam subjected to end pure bending

Load case (2): Pure bending moment M=10					
Mesh	T6	Q8	SBRIE	SBRIEIR	**SBME**
6 x 1	0.9932	0.9919	0.91	0.91	**0.962**
12x1	0.9967	0.9962	0.91	0.91	**1**
Reference solution [18]			1,000 (0.270)		

3.2.Plane flexure of cantilever beam

This structural problem has been widely employed to assess the effectiveness of finite element formulations. In particular, Batoz [21] used this example to evaluate the performance of the CST, LST, Q4, Q4WT [22,23], Q4PS [24], and Q8 elements. Subsequently, Ayad [25] investigated the same benchmark to validate two finite elements, namely FRQ and FRT, developed according to the "Plane Fiber in Rotation" concept. The numerical predictions obtained with various mesh densities are summarized in Table 3.

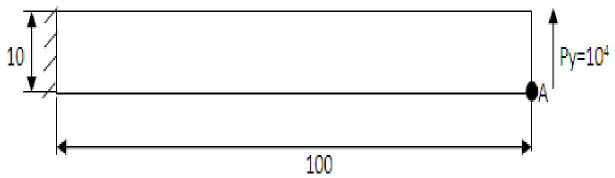

Figure 3. Cantilever beam subjected to uniform vertical load

The numerical values listed in Table III highlight the effectiveness of the proposed formulation. The SBME element produces more accurate predictions than both the FRQ and SBRIEIR elements. Moreover, its performance is

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comparable to that of the Q8 and T6 formulations, despite requiring a smaller number of degrees of freedom, which enhances computational efficiency

TABLE III. Displacement VA of the beam in plane flexure

Elements					
	IE:3HP	EI:3×3	RI:2×2	AI	EI:4×4
Mesh	T6	Q8	FRQ	SBRIEIR	**SBME**
1 x 1	3	3.03	2.76	2.75	**2.83**
2 x 1	-18	-16	(12)**	-12	**-12**
3 x 1	3.7	3.7	3.44	3.43	**3.24**
	-30	-26	-18	-18	**-18**
	3.84	3.84	3.56	3.56	**3.7**
	-42	-36	-24	-24	**-24**
Beam theory [26]	V_A= 4,03				

4.Elastoplastic Analysis

In elastoplastic analysis, the complexity increases due to the fact that plastic strain depends on the material's past stress–strain history. This study utilizes three distinct yield criteria. The Tresca and Von Mises models, which effectively represent the plastic behavior of metals, are examined alongside the Mohr–Coulomb criterion, which is more suitable for materials like concrete, rock, and soil. To simplify the analysis, the viscoplastic method and the initial stress method are employed. The global stiffness matrix, typically derived from elastic assumptions, is calculated only once. The solution is considered converged when the resulting stress state meets the selected yield criteria within defined tolerance limits.

4.1.Bearing capacity analysis of purely coherent soil

The case study examines a deformable strip foundation placed at ground level on a uniform deposit of saturated clay under undrained conditions (see Fig. 4). The soil response is modeled as elastoplastic, with material properties defined by an elastic modulus of E=105kN/m2, a Poisson ratio equal to 0.3, and an undrained shear strength of Cu=100 kN/m2. A distributed load initially set to q=1 kN/m2 is imposed on the foundation and progressively increased throughout the simulation. The computation is performed within a plane-strain framework using a viscoplastic formulation in conjunction with the Von Mises failure criterion. The ultimate state is reached when the contact pressure attains the bearing resistance predicted by Prandtl's theory.he initial problem involves a flexible strip footing resting on the surface of a homogeneous layer of undrained clay, as illustrated in Fig. 4.

$$q_{ultime}=(2+\pi)Cu \tag{12}$$

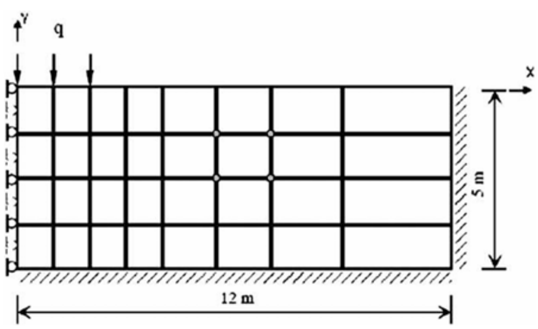

Figure 4. Flexible strip footing : geometry and mesh

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This problem was previously analyzed in [28] using an eight-node quadrilateral finite element.

The numerical results are illustrated in Fig. 5, which shows the variation of the normalized bearing capacity factor q/Cu as a function of the displacement measured along the footing centerline. Examination of the obtained curves indicates that the proposed SBME element achieves fast convergence toward the bearing capacity predicted by Prandtl's solution.

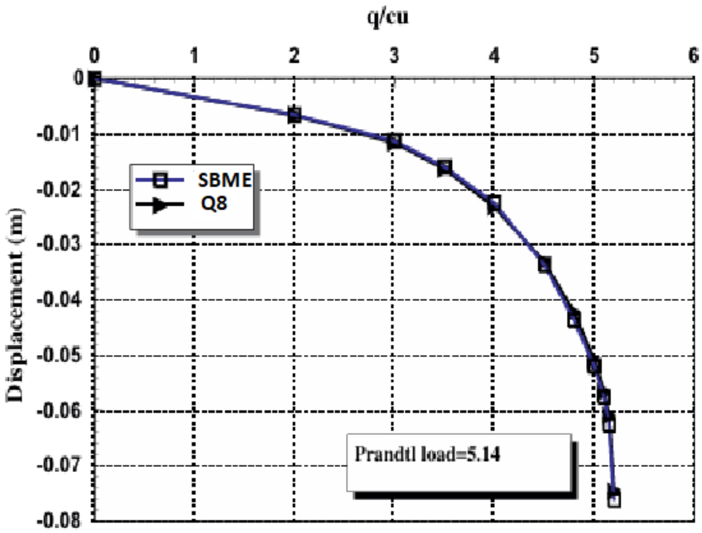

Figure 5. Forced vibration of a rectangular solid ''Displacements versus Time-step"

5.CONCLUSIONS

This study presents the development of a new four-node rectangular membrane finite element, called SBME, based on the strain formulation, designed for both elastic and elastoplastic analyses. The element features a straightforward structure and incorporates higher-order polynomial terms. Numerical results generated with this element show good agreement with theoretical solutions and others numerical experiments. The developed element demonstrates high numerical accuracy in both types of analysis. It exhibits a fast convergence rate toward the reference solutions across all test cases and proves to be well-suited for solving linear and materially nonlinear engineering problems.

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