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Research Article

Modelling of Difference Equations in Single-Input Single-Output and Multi-Input Multi-Output Systems

G. Jayabarathy[1], J. Daphy Louis Lovenia[2*], A.P. Lavanya[3], D. Darling Jemima[4]

[1] Research Scholar, Department of Mathematics, Karunya Institute of Technology and Sciences, Coimbatore, India.jayabarathyg@karunya.edu.in
[2*] Professor, Corresponding Author, Department of Mathematics, Karunya Institute of Technology and Sciences, Coimbatore, India.jdaphy@gmail.com
[3]Assistant Professor, Department of Mathematics, Sri Krishna College of Engineering and Technology, Coimbatore, India.algebralavanya@gmail.com
[4]Assistant Professor, Department of Computer Science and Engineering, Sri Krishna College of Technology, Coimbatore, India.darlingjemimai@gmail.com

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ABSTRACT

The non-linear neutral difference equation of fourth order is given by the form,

$$\Delta^2\left(d(n)\Delta^2\big(y(n)+q(n)y(n-\alpha)\big)\right)+r(n)y(n-\alpha+1)=0$$

where $\{d(n)\}$, $\{q(n)\}$, and $\{r(n)\}$ are positive sequences for $n \in \mathbb{N}$ is studied. Certain new criteria for oscillation of non-linear neutral difference equations for fourth order are developed. The results are applied in Single Input Single Output (SISO) and Multi Input Multi Output (MIMO) systems using Z-transformation. Examples are provided to prove the results.

Keywords: Difference equations, oscillatory behaviour, SISO and MIMO system, Z-transform.

INTRODUCTION

The objective of the paper is to have a clear study of non-linear neutral difference equations of fourth order given by,

$$\Delta^2(d(n)\Delta^2(y(n)+q(n)y(n-\alpha)))+r(n)y(n-\alpha+1)=0 \qquad (1)$$

where $\{d(n)\}$, $\{q(n)\}$, and $\{r(n)\}$ are positive sequences of real numbers, and $d(s)$ satisfies

$$\sum_{s=n_0}^{\infty}\left(\frac{1}{d(s)}\right)=\infty \qquad (2)$$

for $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}$. A non-trivial solution of (1) is said to be oscillatory if the terms of sequence $\{y_n\}$ are neither eventually positive nor eventually negative and non-oscillatory otherwise. The difference equations are used in economics, geometry, electrical networks, biology, etc., see for example [1-10]. In recent times, the difference equations have achieved popularity to study the control systems for continuous and discrete times relating to real life problems. The problems are complex to handle so the control system design SISO & MIMO models to produce appropriate output by reducing the complexity. However usage of SISO proved to be unsatisfactory to solve for multiple systems and hence the reach to multi-input multi-output systems was expanded. For describing the nature of real-life problems the input-output (i/o) model is used. In difference equations i/o is modelled in a more convenient and compact way by transforming the system from one representation into another. In recent times transformation has become a very powerful mathematical tool to solve many problems. The Z-transform is one of the transformations which are widely used in applied mathematics, economics, etc. In this paper, certain results for the oscillatory behaviour in neutral difference equations are developed and are used for constructing SISO & MIMO models by Z-transformation. This process provides a complete outline to study the dynamical behaviour for periodic progressions of time. In Single Input Single Output (SISO) and Multi Input Multi Output (MIMO) system an improved output process is delivered. This paper has been organised as: Methodologies are stated, Definitions with preliminaries are detailed, some Results for oscillatory solutions of (1) are obtained, Examples are given, Mathematical Modelling of (1) by Z transform is shown, followed by conclusion and future work.

OBJECTIVES

The objective is to understand the concept of non-linear neutral difference equations and their applications. To study the oscillatory and non-oscillatory behaviour of fourth order difference equations in control systems using SISO & MIMO models.

METHODS

The oscillatory behaviour of fourth order neutral difference equations are studied using Riccati technique, summation averaging technique, method of summation by parts, comparison and substitution methods. Experimental data or trial data can be evaluated by means of system identification or input-output techniques like eigen-system realization algorithm and least squares estimates. The method of Z transform is used for constructing models of control systems.

RESULTS

Control Systems:

Definition: If $X(s) = X(x_0, x_1, \dots, x_n)$ is the input, and $Y(s) = (y_0, y_1, \dots, y_n)$ is the output then the following equation is formulated.

$$\sum_{s=0}^{n-1} X(s+1) - \sum_{s=0}^{n-1} X(s) = Y(n)$$

A control system is defined as a set of electronic or mechanical device which can regulate another device or another system using control loops. The most essential part of distribution and production in industries takes place in control systems. They play a vital role in various types of industries, automation technology and computerization. Control systems are of 2 types: closed and open loop system. A set with input-output (i/o) system in control system is irreducible if no variable other than zero exist, otherwise the system is said to be reducible**.** The system involves both SISO and MIMO models for gaining output. SISO models are simple process as single parameter is used while MIMO models have comprehensive solutions as they use multiple parameters, thus enhancing the accuracy in stability, efficiency and product quality.

The SISO (Single Input Single Output) Model in Difference Equations:

Classical control system limits itself to SISO model. Here mathematical tool used in SISO is Laplace transformation and primarily analysis is carried out in the frequency domain. Thus the stability, performance, steady-state and transient response of the system are attained, hence their usage in manufacturing and industrial processes are higher. But modern control system employs complex mathematics and so the SISO model is comparatively less used in the industries. The SISO technology can send or receive only one spatial stream at the same time. They are used by inverse Laplace transform in the time domain. Here (1) is called a recursive difference equation. In order to obtain the desired output, firstly the control process is discretized into a number of time-steps. When the initial states x_i at time t_i are identified, from (1) the control signals are determined. The signals are sent to the plant to get y_i at t_i. The same procedure continues in next time-step till the process gets completed.

The MIMO (Multi-Input Multi-Output) Model in Difference Equations:

The MIMO model is used in modern control system. The mathematical tool for the system is linear algebra and the analysis is carried out in time domain. They are used successfully in industries. However, it cannot be used in time-varying systems because of the complexity and the controller designs tend to minor imprecisions in the system. The fourth order difference equations with delay terms play a vital role as they indicate the time delay between input and output in MIMO models. These terms have a great effect on response in time, stability and behaviour in system. Hence essential control is required for gaining accuracy in the models. The controllability and stability are determined when difference equations are introduced for the models. By developing fourth order difference equations, the effects over deviating responses of inputs, and external disturbance are evaluated. Here controllability emphases on system reaction involved in controlling inputs, while stability emphases on ability of the system which returns to equilibrium state after perturbations for gaining preferred outcome. The MIMO model has wireless technology which allows many receivers and transmitters for the transmission of data at a single time interval. The MIMO models supports all wireless products and hence in mobile communications, digital television (DTV), home networks, wireless local area networks (WLANs), etc., they are used. Similar to the SISO model, if mathematical model of controller is chosen at interval t_i, the control process can be operated. The same procedure is followed to gain the output thus having same control principle for both systems.

The Z-Transform: Definition: Here $\{w(p)\}$ be taken as a number sequence for $w(p) = 0$. The Z transformation for the sequence is given by the series,

$$Z(w(p)) = \sum_{p=0}^{\infty} \left(\frac{w(p)}{z^p} \right)$$

where Z is the transform variable.

Modelling of control systems and methods to transform nonlinear difference equations into linear difference equations:

The purpose of mathematical modelling for real world problem is to understand and solve the complexity of problems numerically. There are several methods to solve any system and in this paper results on control systems are discussed. The control systems are concerned with the input-output values which can be determined by the difference equations. The equations are numerically solved and solutions are determined where each solution becomes a control signal. From the signals the output values can be established. In mathematics, analysis and algebra are the core conversion mathematical tool in difference and differential equations. To convert any non-linear equation to linear equation various methods such as the reduction method, solving by chain rule, the logarithmic function, etc., are used. The z- transforms are then applied in linear equations.

Results on Oscillatory behaviour of Difference Equations:

Certain new oscillatory criteria for (1) are established. Beginning with some lemmas followed by theorems, the results are obtained. For every solution $\{y(n)\}$ of (1) the respective sequence $\{z(n)\}$ is given by the form,

$$z(n) = y(n) + q(n)y(n - \alpha) \qquad (3)$$

Lemma 1: Assume $\{y(n)\}$ be a non-negative solution of (1), then exist two cases for (1). Defining for $z(n)$ we have,

(i) $z(n) > 0$, $\Delta z(n) > 0$, $\Delta^2 z(n) > 0$ and $\Delta^3 z(n) > 0$.

(ii) $z(n) > 0$, $\Delta z(n) < 0$, $\Delta^2 z(n) > 0$ and $\Delta^3 z(n) > 0$ for $n \geq n_1 \in N$ where n_1 is large.

Proof: Here $\{y(n)\}$ is a non-negative solution for (1) with every $n \geq n_0$. Here $z(n) > y(n) > 0$ and

$$\Delta^2 (d(n)\Delta^2 z(n)) = -r(n)y(n - \alpha + 1) < 0 \qquad (4)$$

Therefore $p(n)\Delta^2 z(n)$ is of one sign and decreasing. Here $\Delta^3 z(n)$ is also one sign and there are two cases, either $\Delta^3 z(n) < 0$ or $\Delta^3 z(n) > 0$ for $n \geq n_1$ by (4). For $\Delta^3 z(n) < 0$, $\exists$ a constant $D > 0$ such that $p(n)\Delta^2 z(n) \leq -D < 0$. Summing this inequality we get, $\Delta^2 z(n) \leq \Delta^2 z(n_1) - D \sum_{s=n_1}^{n-1} \frac{1}{d(s)}$. If $n \to \infty$ with (2) we have $\Delta z(n) \to -\infty$ then eventually $\Delta z(n) < 0$. But $\Delta^3 z(n) < 0$, $\Delta^2 z(n) < 0$ and $\Delta z(n) < 0$ hence $z(n) < 0$ for every $n \geq n_1$. This gives a contradiction for $\Delta^3 z(n) > 0$ and the proof is completed.

Lemma 2: Assume $\{y(n)\}$ to be a non-negative solution of (1) and $z(n)$ satisfy condition (ii) of Lemma 1. Suppose,

$$\sum_{n_3=n_4}^{\infty} \sum_{n_2=n_3}^{\infty} \frac{1}{d(n_2)} \sum_{n_1=n_2}^{\infty} \left(\sum_{n=n_1}^{\infty} r(n) \right) = \infty \qquad (5)$$

then $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} z(n) = 0$.

Proof: Assume $\{y(n)\}$ to be a non-negative solution of (1). As $z(n) > 0$ and $\Delta z(n) < 0$, then there exist a finite limit, $\lim_{n\to\infty} z(n) = k$. To prove $k = 0$ take $k > 0$, also for $\varepsilon > 0$ there is $k + \varepsilon > z(n) > k$. For $0 < \varepsilon < l\frac{(1-q)}{q}$ the following inequality is verified,

$$y(n) = z(n) - q(n)y(n-\alpha) > l - qz(n-\alpha) > l - q(l+\varepsilon) > kz(n)$$

Here $l = \frac{k-q(k+\varepsilon)}{(k+\varepsilon)} > 0$. With above inequality and (4) we get,

$$\Delta^2\big(d(n)\Delta^2 z(n)\big) \le -r(n)lz(n-\alpha+1)$$

Taking summation for above inequality from $n_1 \to \infty$ and if $n = t$ we have,

$$-\Delta\big(d(n_1)\Delta^2 z(n_1)\big) \ge l\sum_{t=n_1}^{\infty} r(t)z(t-\alpha+1)$$

Taking summation from $n_2 \to \infty$ we get,

$$d(n_2)\Delta^2 z(n_2) \ge l\sum_{n_1=n_2}^{\infty}\left(\sum_{t=n_1}^{\infty} r(t)z(t-\alpha+1)\right)$$

$$\Delta^2 z(n_2) \ge \frac{l}{d(n_2)}\sum_{n_1=n_2}^{\infty}\left(\sum_{t=n_1}^{\infty} r(t)z(t-\alpha+1)\right)$$

when $z(n-\alpha+1) \ge k$, $\Delta^2 z(n_2) \ge \frac{lk}{d(n_2)}\sum_{n_1=n_2}^{\infty}\big(\sum_{t=n_1}^{\infty} r(t)\big)$. Again taking summation $n_3 \to \infty$,

$$-\Delta z(n_3) \ge lk\sum_{n_2=n_3}^{\infty}\frac{1}{d(n_2)}\sum_{n_1=n_2}^{\infty}\left(\sum_{t=n_1}^{\infty} r(t)\right)$$

A final summation from $n_4 \to \infty$ yield,

$$z(n_4) \ge kl\sum_{n_3=n_4}^{\infty}\sum_{n_2=n_3}^{\infty}\frac{1}{d(n_2)}\sum_{n_1=n_2}^{\infty}\left(\sum_{n=n_1}^{\infty} r(t)\right)$$

which contradicts (5) and so $k = 0$. Also from the inequality $0 < y(n) \le z(n)$ we see that $\lim_{n\to\infty} y(n) = 0$ and the proof is completed.

Lemma 3: Assume $m(n) > 0$, $\Delta m(n) \ge 0$, $\Delta^2 m(n) \ge 0$, $\Delta^3 m(n) \le 0$ for every $n \ge n_0$and for every $k \in (0,1)$ $\exists\, N \ge n_0 \ni \frac{m(n-\alpha)}{n-\alpha} \ge \frac{km(n)}{n}$ for every $n \ge N$.

Proof: By mean value theorem and the monotonicity property of $\{\Delta m(n)\}$, we obtain the following result

$$m(n) - m(n-\alpha) = \sum_{s=n-\alpha}^{n-1}\Delta^2 m(s)\alpha \le \Delta^2 m(n-\alpha)$$

then

$$m(n) - m(n-\alpha) = \sum_{s=n-\tau}^{n-1}\Delta^2 m(s)\alpha \le \Delta^2 m(n-\alpha)\alpha$$

Or
$$\frac{m(n)}{m(n-\alpha)} \le 1 + \alpha\frac{\Delta^2 m(n-\alpha)}{m(n-\alpha)} \qquad (4)$$

And also, $m(n-\alpha) \ge m(n-\alpha) - m(n_0) \ge \Delta m(n-\alpha)(n-\alpha-n_0) \ge \Delta^2 m(n-\alpha)(n-\alpha-n_1)$. Hence $k \,\epsilon\, (0,1)$ and $N \ge n_0$ then,

$$\frac{m(n-\alpha)}{\Delta^2 m(n-\alpha)\alpha} \geq l(n-\alpha), \quad n \geq N \tag{5}$$

Combining (4) with (5), we obtain,

$$\frac{m(n)}{m(n-\alpha)} \leq 1+\alpha\frac{1}{l(n-\alpha)} \leq \frac{n}{l(n-\alpha)}$$

Thus proof is completed.

Lemma 4: Let conditions of Theorem 1 hold with $\Delta z(n) \geq 0$ then, $\Delta^4 z(n) \leq 0$ over (N,∞). Then $\frac{z(n+1)}{\Delta^2 z(n)} \geq \frac{n-N}{3}$ for every $n > N$.

Proof: From Mean Value theorem for $\{\Delta^3 z(n)\}$ we get,

$$\Delta z(n) = \Delta z(N) + \sum_{s=N}^{n-1} \Delta^3 z(s) \geq (n-N)\Delta^2\big(\Delta z(s)\big)$$

Taking summation from N to $n-1$, we have,

$$z(n+1) \geq z(n) \geq z(N) + \sum_{s=N}^{n-1} \Delta^2\big(\Delta z(s)\big) = z(N) + (n-N)\Delta^2 z(n) - 2z(n+1) + z(n) - z(N)$$

Hence, $\frac{z(n+1)}{\Delta^2 z(n)} \geq \frac{n-N}{3}$ for $n > N$. Thus the proof is completed.

Lemma 5: Let conditions of Theorem 1 hold with $\Delta z(n) \geq 0$, then $\Delta^4 z(n) \leq 0$ $\forall$ $n \geq N$, thus $(n-N)\frac{\Delta^3 z(n)}{\Delta z(n)} \leq 1$.
Proof: Consider the inequality,

$$\Delta z(n) \geq \Delta^2 \sum_{s=N}^{n-1} \Delta^2 z(s) \geq (n-N)\Delta^3 z(n)$$

In order to the results of oscillation assume the following representations as $n \to \infty$,

$$Q = \lim inf \frac{n}{\Delta d(n)} \sum_{s=n}^{\infty} Q_k(s)\,, \quad R = \lim sup \frac{1}{n}\sum_{s=n}^{\infty} \frac{s}{\Delta d(s)} Q_k(s) \tag{6}$$

where $Q(k(s)) = k(1-q)r(s)\left(\frac{s-\alpha}{s}\right)\left(\frac{s-\alpha-N}{4}\right)$ and $k \in (0,1)$ chosen arbitrarily. Also to satisfy case (i) of Lemma 1 define,

$$b(n) = \Delta d(n)\left(\frac{\Delta^2 z(n)}{\Delta z(n)}\right) \tag{7}$$

and

$$r = \lim_{n\to\infty} \inf n\frac{b(s+1)}{\Delta d(s+1)} \text{ and } T = \lim_{n\to\infty} \sup n\frac{b(s)}{\Delta d(s)} \tag{8}$$

Theorem 1: Assume that condition of Lemma 3 holds and $\{p(n)\}$ is nondecreasing. Let $\{y(n)\}$ be a solution of (1). If

$$Q = \lim inf \frac{n}{\Delta d(n)} \sum_{s=n}^{\infty} Q_k(s) > 1 \tag{8.1}$$

then $\{y(n)\}$ is oscillatory or $y(n) \to 0$ as $n \to \infty$.

Proof: If Lemma 3 holds then conditions (i) and (ii) of Lemma 1 eventually holds. Hence there exist 2 cases: $Q = \infty,\ R = \infty$ and $Q < \infty,\ R < \infty$.

Case I: Let $\{y(n)\}$ be a non-negative solution of (1). Now to prove $\{z(n)\}$, if the contrary $Q \neq \infty,\ R \neq \infty$ is assumed then Q and R do not belong to (i) of Lemma. This is a contradiction for $z(n)$ satisfy (ii) of Lemma 1 for $Q = \infty,\ R = \infty$. Thus from Lemma 2 we get $\lim_{n\to\infty} y(n) = 0$.

Case II: To prove $Q < \infty$ and $R < \infty$ the following conditions are to be satisfied,

$$Q \leq r - r^2\,, \qquad Q + R \leq 1 \qquad \text{(C 1)}$$

Assume $z(n)$ satisfies (i) of Lemma-1. With (1) and Lemma-2 the inequality becomes,

$$y(n) = z(n) - q(n)y(n-\alpha) > z(n) - q(n)z(n-\alpha) \geq \big(1 - q(n)\big)z(n) \geq (1-q)z(n)$$

Using above inequality in (1) then,

$$\Delta^2\big(d(n)\Delta^2 z(n)\big) \leq -r(n)(1-q)z(n-\alpha+1) \leq 0 \qquad (9)$$

From Lemma 5 if $w(n) > 0$ and (9) holds then $w(n)$ satisfy the following,

$$\Delta b(n) = \frac{\Delta\Big(\Delta\big(d(n)\Delta^2 z(n)\big)\Big)}{\Delta z(n)} - \frac{\Delta d(n+1)\Delta^2 z(n+1)}{\big(\Delta z(n+1)\big)^2} - \Delta d(n)\frac{\Delta^2 z(n)}{\Delta z(n)} \qquad (10)$$

$$\leq -r(n)(1-q)\frac{z(n-\alpha+1)}{\Delta z(n)} - b(n+2) - \frac{1}{\Delta d(n+1)}\big(b(n+1)\big)^2$$

For $m(n) = \Delta z(n)$ of Lemma 3 and from (6) we get,

$$\frac{1}{\Delta^2 z(n)} \geq \frac{k(n-\alpha)}{n}\frac{1}{\Delta^2 z(n-\alpha)}, \quad N \leq n$$

From (10),

$$\Delta^2 b(n) \leq -kr(n)\left(\frac{n-\alpha}{n}\right)(1-q)\frac{z(n-\alpha+1)}{\Delta z(n-\alpha)} - b(n+2) - \frac{1}{\Delta d(n+1)}\big(b(n+1)\big)^2$$

Using Lemma 4 for $\frac{z(n+1)}{\Delta^2 z(n)} \geq \frac{n-N}{3}$ then,

$$\Delta b(n) + Q\big(k(n)\big) + b(n+2) + \frac{1}{\Delta d(n+1)}\big(b(n+1)\big)^2 \leq 0 \qquad (11)$$

Since $b(n) > 0,$ then $\Delta^2 b(n) \leq 0$ hence from (11),

$$-\ \Delta b(n) \geq \frac{1}{\Delta d(n+1)}\big(b(n+1)\big)^2 + b(n+2)$$

Summing last inequality,

$$b(n) \leq b(N) + \sum_{s=n}^{n-1} b(s+2)^{-1} \qquad (12)$$

Using (2) in (12) and applying limit gives, $\lim_{n\to\infty} b(n) = 0$. From Lemma 5 and $w(n)$ we get,

$$0 \leq r \leq T \leq 1 \qquad (13)$$

To prove (C 1) holds then there exist $\varepsilon > 0$ and from Q and r of Lemma 5, take $n_2 > N$ such that

$$\frac{1}{\Delta d(n)}\sum_{s=n}^{\infty} Q(l(s)) \geq -\varepsilon + Q \text{ and } \frac{b(n+1)}{\Delta d(n+1)} \geq -\varepsilon + r$$

for every $n \geq n_2$. Summing (11) as $n \to \infty$ and using $\lim_{n\to\infty} b(n) = 0$ then,

$$b(n) \geq \sum_{s=n}^{\infty} Q\big(l(s)\big) + \sum_{s=n}^{\infty}\left(\frac{\Delta d(s+1)b(s+2) + \big(b(s+1)\big)^2}{\Delta d(s+1)}\right) \qquad (14)$$

We know that $\Delta^2 d(n) > 0$ hence,

$$b(n) \geq \sum_{s=n}^{\infty} Q\big(l(s)\big) + \sum_{s=n}^{\infty}\left(\frac{\Delta d(s+1)b(s+2)}{\Delta d(s+1)}\right) + \sum_{s=n}^{\infty}\left(\frac{\big(b(s+1)\big)^2(s+1)\Delta p(s+1)}{\Delta d(s+1)(s+1)\Delta d(s+1)}\right)$$

$$b(n) \geq (Q-\varepsilon) + \sum_{s=n}^{\infty}\frac{b(s+1)}{\Delta d(s+1)}\left(\frac{b(s+1)(s+1)\Delta d(s+1)}{(s+1)\Delta d(s+1)}\right) + \sum_{s=n}^{\infty}\left(\frac{b(s+2)(s+1)\Delta d(s+1)}{(s+1)\Delta d(s+1)}\right)$$

$$\geq (-\varepsilon+Q) + (-\varepsilon+r)^2\sum_{s=n}^{\infty}\left(\frac{(s+1)\Delta d(s+1)}{s+1}\right) + \sum_{s=n}^{\infty} b(s+2)$$

$$n\frac{b(n)}{\Delta d(n)} \geq (-\varepsilon+Q) + (-\varepsilon+r)^2 n\sum_{s=n}^{\infty}\frac{\Delta d(s+1)}{\Delta d(s)} + n\frac{b(s+2)}{\Delta d(s)}$$

With (1),

$$n\frac{b(n)}{\Delta d(n)} \geq (-\varepsilon+Q) + (-\varepsilon+r)^2 \qquad (15)$$

Taking $\lim\ inf$ on both sides as $n \to \infty$ we obtain get, $\lim_{n\to\infty} inf\, n\frac{b(n)}{\Delta d(n)} \geq (-\varepsilon+Q) + (-\varepsilon+r)^2$. By repeating the same we have, $r \geq (-\varepsilon+Q) + (-\varepsilon+r)^2$. Here $\varepsilon > 0$ is arbitrarily small then from $r \geq Q + r^2$ the following result is obtained.

$$Q \leq r - r^2 \qquad (16)$$

Thus the inequality of (C 1) is proved. Now to prove the second inequality multiply (11) by $\frac{n}{\Delta d(n)}$. Taking summation from n_2 to ∞ we get,

$$\sum_{s=n_2}^{\infty}\frac{s}{\Delta d(s)}\Delta b(s) \leq -\sum_{s=n_2}^{\infty}\frac{s}{\Delta d(s)}Q\big(k(s)\big) - \sum_{s=n_2}^{\infty}\frac{s}{\Delta d(s)}\left(b(s+2) + \frac{\big(b(s+1)\big)^2}{\Delta d(s+1)}\right)$$

By summation by parts, we obtain

$$\frac{nb(n)}{\Delta d(n)} \leq \frac{n_2 b(n_2)}{\Delta d(n_2)} - \sum_{s=n_2}^{\infty}\frac{s}{\Delta d(s)}Q\big(k(s)\big) - \sum_{s=n_2}^{\infty}\frac{s}{\Delta d(s)}\left(b(s+2) + \frac{\big(b(s+1)\big)^2}{\Delta d(s+1)}\right) + \sum_{s=n_2}^{\infty}\frac{\Delta(s)}{\Delta d(s)}b(s+1)$$

As $\Delta d(n) > 0$ then, $\frac{\Delta(s)}{\Delta d(s)} = \frac{\Delta^2(s)}{\Delta d(s+1)} - \frac{\Delta(s)\Delta d(s)}{\Delta d(s+1)} \leq \frac{(s+1)}{\Delta d(s+1)}$. Therefore,

$$\frac{nb(n)}{\Delta d(n)} \leq \frac{n_2 b(n_2)}{\Delta d(n_2)} - \sum_{s=n_2}^{\infty}\frac{s}{\Delta d(s)}Q\big(k(s)\big) + \sum_{s=n_2}^{\infty}\left[\frac{(s+1)}{\Delta d(s+1)}b(s+1) - \frac{(s)b(s+2)}{\Delta d(s)} - \frac{(s)\big(b(s+1)\big)^2}{\Delta d(s)\Delta d(s+1)}\right]$$

$$\frac{nb(n)}{\Delta d(n)} \leq \frac{n_2 b(n_2)}{\Delta d(n_2)} - \sum_{s=n_2}^{\infty}\frac{s}{\Delta d(s)}Q\big(k(s)\big) + \sum_{s=n_2}^{\infty}\left[\frac{(s+1)}{\Delta d(s+1)}w(s+1) - \frac{(s)\big(b(s+1)\big)^2}{\Delta d(s)\Delta d(s+1)} - \frac{(s)b(s+2)}{\Delta d(s)}\right]$$

By means of the inequality,

$$uA - \frac{D}{C}\left(\frac{A^2}{C+1} - B\right) \leq \frac{u}{D}(C+1)$$

where $u = \frac{(s+1)}{\Delta d(s+1)},\ A = b(s+1),\ B = b(s+2),\ C = \Delta p(s+1), C = \Delta p(s), D = S$ we get the following,

$$\frac{nb(n)}{\Delta d(n)} \leq \frac{n_2 b(n_2)}{\Delta d(n_2)} - \sum_{s=n_2}^{\infty} \frac{s}{\Delta d(s)} Q(k(s)) + \sum_{s=n_2}^{\infty} \left(\frac{s+1}{s}\right)$$

It follows that

$$\frac{nb(n)}{\Delta d(n)} \leq \frac{n_2 b(n_2)}{\Delta d(n_2)} - \frac{1}{n}\sum_{s=n_2}^{\infty} \frac{s}{\Delta d(s)} Q(k(s)) + \frac{1}{n}\sum_{s=n_2}^{\infty} \left(\frac{s+1}{s}\right)$$

Using lim sup from n to ∞ then, $T + R \leq 1$. From (13) and (16),

$$Q \leq -r^2 + r \leq T \leq 1 - R$$

Therefore the second inequality of (C 1) is also proved and thus both the cases are satisfied. Here case I hold but the result obtained from case II gives a contradiction to (8.1). Thus the proof is completed.

Theorem 2: Assume the condition of Lemma 2 holds. Let $\{d(n)\}$ be increasing, $\{y(n)\}$ be a solution of (1). If

$$Q + R > 1 \tag{17}$$

then $\{y(n)\}$ either oscillate or satisfy the condition $\lim_{n\to\infty} y(n) = 0$.

Proof: Assume $\{d(n)\}$ to be a non-oscillatory solution in (1). If case I for Theorem 1 hold, $z(n)$ do not satisfy case (i) for Lemma 1. It is known that $z(n)$ eventually satisfies case (ii) of Lemma 1 which is similar to proof for case I in Theorem 1 hence $\lim_{n\to\infty} y(n) = 0$ by Lemma 2. Without loss of generality case II of Theorem 1 is assumed and then the same result, $\lim_{n\to\infty} y(n) = 0$ by Lemma 2 is achieved. If Q and R satisfy the inequality,

$$Q + R \leq 1$$

which yield a contradiction to (17). Thus the proof is completed.

Examples: The examples demonstrate the main results.

Example 1: The difference equation of the following form is considered

$$\Delta^2\left(n\Delta^2(y(n+1))\right) = \frac{(-1)n^5}{(n+1)(n+2)(n+3)(n+4)}\left[38y_n^5 + 26y_n^4 + 153y_n^3 + 122y_n^2 + 38y_n + 4\right]$$

Thus every condition of Theorem 1 is satisfied. Here $\{y_n\} = \left\{\frac{1}{n}\right\}$ is one such solution and every solution is oscillatory.

Example 2: The nonlinear difference equation of the following form is considered

$$\left(w(p+1)\right)^2 - 5\left(w(p+1)\right)wp - 6wp^2 = 0 \qquad (E1)$$

The Z transform converts nonlinear difference equations to linear difference equations. Hence take

$$u = \frac{w(p+1)}{w_p} \qquad (E2)$$

Substituting in *(E1)* implies $u^2 - 5u - 6 = 0$,

$$u = 6 \qquad (E3) \qquad \text{Or} \qquad u = -1 \qquad (E4)$$

By *(E3)* and *(E4)* in *(E2)* implies,

$$6 = \frac{w(p+1)}{w_p} \qquad \text{(or)} \qquad -1 = \frac{w(p+1)}{w_p}$$

This implies,

$$w(p+1) - 6w(p) = 0 \quad (E5) \text{ Or } w(p+1) + w(p) = 0 \quad (E6)$$

Here *(E5)* and *(E6) become* linear equations. Now taking Z-transform for *(E5)*, we get,

$$Z\{w(p+1)\} - 6Z\{w(p)\} = 0$$

where $w_0 = 1$, then

$$zZ\{w(p)\} - zw_0 - 6Z\{w(p)\} = 0$$

This implies $Z\{w(p)\}(z-6) = z$ which gives $\{w(p)\} = \frac{z}{z(z-6)}$, thus $w(p) = 6^p$. Similarly, $-1 = \frac{w(p+1)}{w(p)}$ gives $w(p+1) + w(p) = 0$ then

$$Z\{w(p+1)\} + Z\{w(p)\} = 0 \qquad (E7)$$

Taking Z-transforms for $(E7)$, $zZ\{w(p)\} - zw_0 + Z\{w(p)\} = 0$ implies $\{w(p)\} = \frac{z}{z(z+1)}$ thus obtain the result, $w(p) = (-1)^p$.

Mathematical Modelling of Difference Equations in Control Systems with Z-transform:

The mathematical modelling plays a vital role in resolving the existing problems. Any system can be represented mathematically and with existing mathematical tool the necessary conditions and solutions can be determined. The main purpose of this mathematical modelling is to convert a non-linear system to a linear system of equations. Numerical analysis, computational methods, iterations, etc., can solve the problem numerically. In this paper, the Z transformation is used for converting difference equations from non-linear form to linear form and also for solving the linear problems numerically. Commonly, the SISO models are linear system of equations from which the roots for characteristic equations should not be non-distinct. In terms of MIMO models both non-linear and linear system of equations are used from which the control signals are explicitly expressed. For a linear system, the solution

$$y(n) = \begin{cases} 1 & \text{for n} \geq 0 \\ 0 & \text{for n} < 0 \end{cases}$$

is called as a unit step of difference equations. Here the solution $y(n)$ is called as discrete functions or as control signals. For example if $y(n) = \left(\frac{1}{2}\right)^n$ then control signal oscillate for sequence, $y(n) = x(n+1) - x(n)$ and $y(n) = x(n) - x(n-1)$ is shown.

CONCLUSION

In this paper, modelling of difference equations for SISO and MIMO systems provides an understanding on technical approaches. The control system is studied and its types were discussed with examples. New conditions and criteria for oscillation of difference equations by Z-transform for SISO and MIMO models are established. The methodology to transform difference equations of nonlinear form into linear form is analysed. The procedures to apply Z-transforms for linear difference equations are shown with example.

FUTURE WORK

The future work is to study the properties such as stability, bifurcations, etc., for SISO and MIMO models of control systems using algorithms using higher order difference equations. Also to extend the work for refining models, integrate advance techniques and control strategies, thus applying them to large scale and complex systems of real life application. These works enhance the understanding and importance of difference equations and their applications in SISO and MIMO systems which eventually lead to efficient solutions of real-world problems.

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