Journal of Information Systems Engineering and Management
2025, 10(16s)
e-ISSN: 2468-4376
https://www.jisem-journal.com/
Research Article

Intuitionistic Fuzzy Fixed Point Theory: Relation-Theoretic Approaches and Applications

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ARTICLE INFO

Received: 05 Dec 2024
Revised: 26 Jan 2025
Accepted: 10 Feb 2025

ABSTRACT

This paper explores the concept of intuitionistic fuzzy R-φ- contractive mappings and establishes key results regarding the existence and uniqueness of fixed points in non-Archimedean intuitionistic fuzzy metric spaces. To illustrate the applicability of these findings, several examples are provided. Additionally, the main results are utilized to demonstrate the existence and uniqueness of solutions for Caputo fractional differential equations within the framework of intuitionistic fuzzy metric spaces.

Keywords: Intuitionistic fuzzy metric spaces; fixed points; binary relations; R-φ- contractive mappings; Caputo fractional differential equations.

MSC: 47H10; 54H25

1. Introduction:

The concept of fuzzy sets, introduced by Zadeh [24] in 1965, has transformed various scientific and engineering applications. Fuzzy sets are functions from a non-empty set X to [0, 1], enabling the modeling of uncertainty and imprecision. Building upon this concept, Kramosil and Michalek [9] introduced fuzzy metric spaces, which were later modified by George and Veeramani [5] to include the Hausdorffness property.

The fuzzy fixed point theory, initiated by Grabiec [7] in 1988, has undergone significant developments. Grabiec introduced G-Cauchy sequences and G-complete fuzzy metric spaces, providing a fuzzy metric version of Banach's contraction principle. Subsequent research has led to numerous fixed point results in fuzzy metric spaces.

However, the concept of G-completeness has been found to be somewhat restrictive, as even the set of real numbers is not complete in this sense. To address this limitation, George and Veeramani [5] modified the definition of fuzzy metric spaces and M-Cauchy sequences, introducing a Hausdorff topology in their new framework. Extending the foundational work of Gregori and Sapena [8], Mihet [14] made significant contributions to the field in 2008. Specifically, Mihet expanded the scope of fuzzy contractive mappings and established a fuzzy Banach contraction principle for complete non-Archimedean fuzzy metric spaces, adhering to the framework developed by Kramosil and Michalek.

In recent years, researchers have explored the intersection of fuzzy metric spaces and relation-theoretic fixed point theory. Turinici [23] initiated this line of inquiry, which gained momentum with the contributions of Ran and Reurings [19] and Nieto and Lopez [15,16]. These authors equipped the contractive condition with an ordered binary relation, providing new versions of Banach's contraction principle.

This research builds upon the foundational work presented in the paper by Samera M. Saleh, Waleed M. Alfaqih *et. al* [22] in 2022. The authors' innovative relation-theoretic fixed point theorems in fuzzy metric spaces have inspired our investigation into extending these results to the more general framework of intuitionistic fuzzy metric spaces. By leveraging their insights and methodologies, we aim to provide new theoretical results and applications in the context of intuitionistic fuzzy metric spaces.

This paper extends the existing results on fuzzy metric spaces to the more general framework of intuitionistic fuzzy metric spaces. By incorporating both membership and non-membership functions, intuitionistic fuzzy metric spaces offer a more refine and flexible approach to modeling uncertainty. Our research builds upon the foundation established by the aforementioned authors, providing new insights and results in the context of intuitionistic fuzzy metric spaces.

2. Preliminaries:

"**Definition 1([9]):** A continuous t-norm $*$ is a continuous binary operation $*: [0,1] \times [0,1] \to [0,1]$ which is commutative and associative and it satisfies the following properties:

(i) $t * 1 = t \quad \forall t \in [0,1]$.

(ii) $t * s \leq u * v$ whenever $t \leq u$ and $s \leq v\ \forall t, s, u, v \in [0,1]$.

Some well-known examples of continuous t-norm include: $t * s = min\{t, s\}$, $t * s = ts$, and $t * s = max\{t + s - 1\}, \forall t, s \in [0,1]$.

Kramosil and Michalek [9] defined fuzzy metric spaces as under.

Definition 2([9]): Consider M as a fuzzy set on $X^2 \times [0, \infty)$ and $*$ a continuous t-norm. Assume that $\forall\, x, y, z \in X$ and $t, s > 0$:

(KMi) $M(x, y, 0) = 0$.

$(KMii)$ $M(x, y, t) = 1\ iff\ x = y$.

$(KMiii)$ $M(x, y, t) = M(y, x, t)$.

$(KMiv)$ $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$.

(KMv) $M(x, y, .): [0, \infty) \to [0,1]$ is left continuous.

Then $(X, M,*)$ is called a fuzzy metric space (Kramosil and Michalek's sense).

Definition 3([14]): Consider M as a fuzzy set on $X^2 \times [0, \infty)$ and $*$ a continuous t-norm. Assume that $\forall\, x, y, z \in X$ and $t, s > 0$:

(NMi) $M(x, y, 0) = 0$.

$(NMii)$ $M(x, y, t) = 1\ iff\ x = y$.

$(NMiii)$ $M(x, y, t) = M(y, x, t)$.

$(NMiv)$ $M(x, y, t) * M(y, z, s) \leq M(x, z, max\{t, s\})$.

(NMv) $M(x, y, .): [0, \infty) \to [0,1]$ is left continuous.

Then $(X, M,*)$ is called a Non-Archimedean fuzzy metric space.

It can be verified that the triangular inequality (NM-iv)o implies (KM-iv). This indicates that every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

In general, the topology of a fuzzy metric space, as defined by Kramosil and Michalek, is not Hausdorff. To address this, George and Veeramani [5,6] made slight modifications to the definition of fuzzy metric spaces, ensuring that the topology of the newly defined fuzzy metric space is Hausdorff.

Definition 4([5,6]): Consider M as a fuzzy set on $X^2 \times [0, \infty)$ and $*$ a continuous t-norm. Assume that $\forall\, x, y, z \in X$ and $t, s > 0$:

(GVi) $M(x, y, 0) > 0$.

$(GVii)$ $M(x, y, t) = 1\ iff\ x = y$.

$(GViii)$ $M(x, y, t) = M(y, x, t)$.

$(GViv)$ $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$.

(GVv) $M(x,y,.) : (0,\infty) \rightarrow [0,1]$ is left continuous.

Then $(X, M,*)$ is defined as a fuzzy metric space in accordance with George and Veeramani's concept.

Remark 1 ([5]): The topology associated with a fuzzy metric space, as defined in Definition 4, is Hausdorff.

Remark 2 ([5]): Each fuzzy metric space as defined in Definition 4 is also a fuzzy metric space according to Definition 2, though the reverse is not generally true.

Remark 3 ([9]): For all $x, y \in X$, $M(x,y,.)$ is a non-decreasing mapping.

Definition 5 ([5,6,7]): Let $M(x,y,*)$ be a fuzzy metric space. A sequence $\{x_n\} \in X$ is said to be

i. Convergent to $x \in X$ if $\lim_{n\to\infty} M(x,y,t) = 1 \quad \forall t > 0$, in this case we write $\lim_{n\to\infty} x_n = x$.
ii. Cauchy if $\forall \varepsilon > 0$ and $t > 0, \exists N \in \mathbb{N}$ satisfying $M(x_n, x_{n+p}, t) > 1 - \varepsilon, \forall n \geq N$ and $p \in \mathbb{N}_0$.

Let $(X, M,*)$ be a fuzzy metric space. If every Cauchy sequence in X is convergent in X , then X is said to be complete.

Lemma 1 ([19]): If $M(x,y,*)$ is a fuzzy metric space, then M is a continuous function on $X^2 \times (0,\infty)$.

Definition 6([18]): If $M(x,y,*)$ is a fuzzy metric space. Then the mapping M is said to be continuous function on $X^2 \times (0,\infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x,y,t)$$

Whenever $\{(x_n, y_n, t_n)\}$ is sequence in $X^2 \times (0,\infty)$ which converges to a point $(x,y,t) \in X^2 \times (0,\infty)$, i.e.

$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1$ and $\lim_{n\to\infty} M(x, y, t_n) = M(x,y,t)$.

Roldán-López-de-Hierro [19] defined a comparison function ψ: [0, 1] → [0, 1] which satisfies

A. The function ψ is non-decreasing and left continuous.
B. $\psi(t) < t$ for all $t \in (0,1)$;
C. $\psi(0) = 0$.

Let Ψ represent the set of all such functions ψ.

For instance, consider $\Psi(t) = t^2$ for all $t \in [0,1]$. It is important to note that, according to the previous definition, the condition $\psi(1) = 1$ does not necessarily hold true.

Remark 4([21]): Let $\psi \in \Psi$.

(i) $\psi(t) \leq t, \ \forall t \in [0,1]$.
(ii) If $\psi(t_0) = t_0$ for some $t_0 \in (0,1]$, then $t_0 = 1$.
(iii) If $\{t_n\} \subset [0,1]$ and $\psi(t_n) \rightarrow 1$, then $t_n \rightarrow 1$.

Definition 7([17]): Let M and N be fuzzy sets on $X^2 \times (0,\infty)$, * is a be a continuous t-norm and ◊ is a continuous t-conorm. If M and N satisfy the following conditions we say that (M, N) is intuitionistic fuzzy metric on X:

$(IFM1)$ $M(x,y,t) + N(x,y,t) \leq 1$

$(IFM2)$ $M(x,y,t) > 0$

$(IFM3)$ $M(x,y,t) = 1$ if and only if $x = y$.
$(IFM4)$ $M(x,y,t) = M(y,x,t)$
$(IFM5)$ $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$
$(IFM6)$ $M(x,y,.) : (0,\infty) \rightarrow (0,1]$ is continuous.
$(IFM7)$ $N(x,y,t) < 1$
$(IFM8)$ $N(x,y,t) = 0$ if and only if $x = y$.
$(IFM9)$ $N(x,y,t) = N(y,x,t)$
$(IFM10)$ $N(x,y,t) \lozenge N(y,z,s) \geq N(x,z,t+s)$
$(IFM11)$ $N(x,y,.) : (0,\infty) \rightarrow (0,1]$ is continuous

A five tuple $(\mathrm{X}, \mathrm{M}, \mathrm{N}, *, \lozenge)$ intuitionistic fuzzy metric space. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Definition 8([17]): Let $(\mathrm{X}, \mathrm{M}, \mathrm{N}, *, \lozenge)$ be an intuitionistic fuzzy metric space and $t > 0$, $r \in (0,1)$ and $x \in X$. The set $B_x(r,t)$

$$B_x(r,t) = \{y \in X : M(x,y,t) > 1 - r,\ N(x,y,t) < r\}$$

The set $B_x(r,t)$ is said to be an open ball with center x, radius r with respect to t.

Definition 9([17]): Let $(\mathrm{X}, \mathrm{M}, \mathrm{N}, *, \lozenge)$ be an intuitionistic fuzzy metric space and $\{x_n\} \in X$ be sequence

i. $\{x_n\}$ is called convergent to x if for all $t > 0$ and $r \in (0,1)$ there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r, N(x_n, x, t) > r$ for all $n \geq n_0$. ($M(x_n,x,t) \to 1$ &$N(x_n,x,t) \to 0$ $as\ n \to \infty\ for\ each\ t > 0$).
It is denoted by $x_n \to x$ as $n \to \infty$.
ii. $\{x_n\}$ is called Cauchy sequence if for $t > 0$ and $r \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r, N(x_n, x_m, t) > r$ for all $n, m \geq n_0$.
iii. $(\mathrm{X}, \mathrm{M}, \mathrm{N}, *, \lozenge)$ is called (M,N) complete if every Cauchy sequence is convergent.

Definition 10: Let M and N be fuzzy sets on $X^2 \times (0,\infty)$, * is a be a continuous t-norm and $\diamond$ is a continuous t-conorm. If M and N satisfy the following conditions we say that (M,N) is non- archimedean intuitionistic fuzzy metric on X:

$(IFM1)$ $M(x,y,t) + N(x,y,t) \leq 1$

$(IFM2)$ $M(x,y,t) > 0$

$(IFM3)$ $M(x,y,t) = 1$ if and only if $x = y$.
$(IFM4)$ $M(x,y,t) = M(y,x,t)$
$(IFM5)$ $M(x,y,t) * M(y,z,s) \leq M(x,z,max\{t,s\})$
$(IFM6)$ $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous.
$(IFM7)$ $N(x,y,t) < 1$
$(IFM8)$ $N(x,y,t) = 0$ if and only if $x = y$.
$(IFM9)$ $N(x,y,t) = N(y,x,t)$
$(IFM10)$ $N(x,y,t) \lozenge N(y,z,s) \geq N(x,z,min\{t,s\})$
$(IFM11)$$N(x,y,.) : (0,\infty) \to (0,1]$ is continuous

A five tuple $(\mathrm{X}, \mathrm{M}, \mathrm{N}, *, \lozenge)$ non-archimedean intuitionistic fuzzy metric space. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Here, we recall some fundamental concepts in relation theory.

Definition 11([12]): A subset $\mathfrak{R}$ of X^2 is defined as a binary relation on X. If $(x,y) \in \mathfrak{R}$ (alternatively, we may write $x\mathfrak{R}y$ instead of $(x,y) \in \mathfrak{R}$, then we state that "x is related to y under $\mathfrak{R}$". If either $x\mathfrak{R}y$ or $y\mathfrak{R}x$, we denote this as $[x,y] \in \mathfrak{R}$.

Note that X^2 is a binary relation on X known as the universal relation. In this context, X refers to a non-empty set, and $\mathfrak{R}$ denotes a non-empty binary relation on X.

Definition 12([12,13]): A binary relation $\mathfrak{R}$ on a non-empty set X is characterized as follows

(i) Reflexive if $x\mathfrak{R}x$, $\forall x \in X$.
(ii) Transitive if $x\mathfrak{R}y$ and $y\mathfrak{R}z$ imply $x\mathfrak{R}z$, $\forall x,y,z \in X$.
(iii) Antisymmetric if $x\mathfrak{R}y$ and $y\mathfrak{R}x$ imply $x = y, \forall x,y \in X$.
(iv) partial order if it is reflexive, antisymmetric and transitive.
(v) complete if $[x,y] \in \mathfrak{R}$ $\forall x,y \in X$.
(vi) f -closed if $(x,y) \in \mathfrak{R} \Rightarrow (fx,fy) \in \mathfrak{R}, \forall x,y \in X$, where $f: X \to X$ is a mapping.

Definition 13([2]): Consider X as a non-empty set and $\mathfrak{R}$ as a binary relation on X. A sequence $\{x_n\} \subset X$ is termed an $\mathfrak{R}$-preserving sequence if $(x_n, x_{n+1}) \in \mathfrak{R}, \forall n \in \mathbb{N}$.

In a recent work, Alfaqih *et al.* [3] introduced a relation-theoretic perspective for the fuzzy version of the Banach contractive principle. The authors developed relation-theoretic variations of several fuzzy metrical concepts as described below.

Definition 14([3]): A binary relation $\mathfrak{R}$ on X is considered M-self-closed if for any convergent $\mathfrak{R}$-preserving sequence $\{x_n\} \subset X$ that converges (in the fuzzy sense) to some $x \in X$, there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that $(x_{n_k}, x) \in \mathfrak{R}$.

Example 1([3]): Let $X = (0,4]$ and $*$ be the product t-norm defined by $t * s = ts,\ \forall t, s \in [0,1]$. Define M for all $x, y \in X$ and $t > 0$.

$$M(x,y,t) = \begin{cases} 0, & if\ t = 0 \\ \dfrac{2t}{2t + |x-y|}, & if\ t \neq 0 \end{cases}$$

Define $\mathfrak{R}$ on X as follows

$$\mathfrak{R} = \{(1,1), (1,2), (2,1), (2,2), (1,4), (2,4)\}$$

Note that if $\{x_n\}$ is an $\mathfrak{R}$-preserving sequence converging to some $x \in X$, then there exists $N \in \mathbb{N}$ such that either $x_n = 1,\ \forall n \geq N$ or $x_n = 2,\ \forall n \geq N$. Consequently, $\{x_{N+i}\}_{i \in N}$ is a subsequence of $\{x_n\}$ with $x_{N+i}\mathfrak{R}x$ for each $i \in \mathbb{N}$. Thus, $\mathfrak{R}$ is M-self closed.

Definition 15: A sequence $\{x_n\}$ is termed $\mathfrak{R}$-Cauchy if $x_n\mathfrak{R}x_{n+1}$ for all $n \in \mathbb{N}_0$ and for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall t > 0$

$$M(x_n, x_{n+\mathfrak{p}}, t) > 1 - \varepsilon, \qquad \forall n \geq N, \mathfrak{p} > N_0$$

Remark 5: Any given Cauchy sequence can be considered an $\mathfrak{R}$-Cauchy sequence for any arbitrary binary relation $\mathfrak{R}$. The concepts of $\mathfrak{R}$-Cauchyness and Cauchyness align when $\mathfrak{R}$ is defined as the universal relation.

Definition 16([22]): A fuzzy metric space (X, M, $*$) with a binary relation $\mathfrak{R}$ is described as $\mathfrak{R}$-complete if each $\mathfrak{R}$-Cauchy sequence converges within X.

Remark 6: Any complete fuzzy metric space is also an $\mathfrak{R}$-complete fuzzy metric space for any arbitrary binary relation $\mathfrak{R}$. When $\mathfrak{R}$ is considered as the universal relation, $\mathfrak{R}$-completeness and completeness are equivalent."

This paper introduces the concept of intuitionistic fuzzy $\mathfrak{R} - \psi$ contractive mappings and presents several significant findings regarding the existence and uniqueness of fixed points for these mappings within the framework of non-Archimedean intuitionistic fuzzy metric spaces (as defined by Kramosil and Michalek as well as George and Veeramani). These findings extend and generalize the results from previous works [6,19]. Additionally, we provide illustrative examples to support our findings. In the final section, we apply our fixed point results to establish the existence and uniqueness of solutions for Caputo fractional differential equations.

3. Main Results:

Our primary section begins with a lemma that will be crucial in proving our main results.

Lemma 2: Consider a function $h: I \to I$ and transitive binary relation $\mathfrak{R}$ that is h-closed. Suppose there exists an $q_0 \in I$ such that $q_0\mathfrak{R}hq_0$ and define a sequence $\{q_n\} \in I$ by $q_n = hq_{n-1}, \forall n \geq N_0$. Then

$$q_m\mathfrak{R}q_n \quad \forall m, n \in N_0 \text{ with } m < n. \qquad (1)$$

Definition 17: Consider (I, P, Q, τ, μ) as a non-Archimedean intuitionistic fuzzy metric space. Let $\mathfrak{R}$ be a binary relation on I and $h: I \to I$. We define h as a intuitionistic fuzzy $\mathfrak{R} - \varphi$ contractive mapping if there exist $\varphi \in \Omega$ such that $\forall\ q, p \in I, t > 0$ with $q\mathfrak{R}p$.

$$P(q,p,t) > 0 \Rightarrow min\{P(q,p,t), max\{P(hq,q,t), P(p,hp,t)\}\} \leq \varphi(P(hq,hp,t)). \qquad (2)$$

$$Q(q,p,t) < 1 \Rightarrow max\{Q(q,p,t), min\{Q(hq,q,t), Q(p,hp,t)\}\} \geq \varphi\big(Q(hq,hp,t)\big). \quad (3)$$

This illustrates a KM-fuzzy R – ψ–contractive mapping.

Example 2: Consider $I = [0,\infty)$ and let τ be the product t-norm given by $\tau(t,s) = ts$ and t-conorm $\mu(t,s) = t + s - ts, \ \forall t,s \in [0,1]$. Define

$$P(q,p,t) = \begin{cases} 0 & if\ t = 0 \\ \left(\frac{t}{1+t}\right)|q-p| & if\ t \neq 0 \end{cases}$$

$$Q(q,p,t) = \begin{cases} 1 & if\ t = 0 \\ \left(\frac{1}{1+t}\right)|q-p| & if\ t \neq 0 \end{cases}$$

Let $h: I \to I$ defined as

$$h = \begin{cases} q/3, & if\ q \in [0,3] \\ q/2, & if\ q \in (3,\infty) \end{cases}$$

Define binary relation $\mathfrak{R}$ on I as $q\mathfrak{R}p \Leftrightarrow q,p \in [0,3], q \leq p$ and $\varphi: [0,1] \to [0,1]$ by $\varphi(t) = t^2$.

Then h is intuitionistic fuzzy $\mathfrak{R} - \varphi -$ contractive mapping.

We are now prepared to present and demonstrate our primary result as follows.

Theorem 1: Consider (I,P,Q,τ,μ) to be a non-Archimedean intuitionistic fuzzy metric space with a binary relation $\mathfrak{R}$ and a mapping $h: I \to I$. Suppose I is $\mathfrak{R}$- complete and h is intuitionistic fuzzy $\mathfrak{R} - \varphi -$ contractive mapping such that:

i. There is $q_0 \in I$ such that q_0 is related to $h(q_0)$ by $\mathfrak{R}$ and $P(q_0, hq_0, t) > 0$ and $Q(q_0, hq_0, t) < 1$ for all $t > 0$.
ii. Relation $\mathfrak{R}$ is transitive and closed under h.
iii. One of the following conditions is true:
    a. The function h exhibits continuity or
    b. Relation $\mathfrak{R}$ is P- self closed and Q-self closed.

Consequently, h possesses a fixed point within I.

Proof: We can find $q_0 \in I$ from (i) such that q_0 is related to $h(q_0)$ by $\mathfrak{R}$ and $P(q_0, hq_0, t) > 0$ and $Q(q_0, hq_0, t) < 1$ for all $t > 0$. Let a sequence $\{q_0\}$ in I where $h(q_n) = q_{n+1}, \ \forall n \in N_0$. If $q_n = q_{n+1}$ for some $n \in N_0$, then q_n is fixed point of h. Suppose $q_n \neq q_{n+1}$ for all $n \in N_0$.

As $P(q_0, hq_0, t) = P(q_0, q_1, t) > 0, \forall t > 0$. Given Lemma 2 and equation (2), we deduce

$$min\{P(q_0,q_1,t), max\{P(hq_0,q_0,t), P(q_1,hq_1,t)\}\} \leq \varphi(P(hq_0,hq_1,t))$$

$$\Rightarrow \quad min\{P(q_0,q_1,t), max\{P(q_1,q_0,t), P(q_1,q_2,t)\}\} \leq \varphi\big(P(q_1,q_2,t)\big) \quad (4)$$

$$\Rightarrow \quad 0 < P(q_0,q_1,t) \leq \varphi(P(q_1,q_2,t)) \leq P(q_1,q_2,t)$$

Similarly, As $Q(q_0, hq_0, t) = Q(q_0, q_1, t) < 1, \forall t > 0$. Given Lemma 2 and equation (3), we deduce

$$max\{Q(q_0,q_1,t), min\{Q(hq_0,q_0,t), Q(q_1,hq_1,t)\}\} \geq \varphi(Q(hq_0,hq_1,t))$$

$$\Rightarrow \quad max\{Q(q_0,q_1,t), min\{Q(q_1,q_0,t), Q(q_1,q_2,t)\}\} \geq \varphi\big(Q(q_1,q_2,t)\big) \quad (5)$$

$$\Rightarrow \quad 0 < Q(q_0,q_1,t) \geq \varphi(Q(q_1,q_2,t)) \geq Q(q_1,q_2,t)$$

If there exists a $t_0 > 0$ for which $P(q_1,q_2,t_0) = 0$, then $\varphi\big(P(q_1,q_2,t_0)\big) = 0$. This indicates that $P(q_1,q_2,t_0) = 0$ (as a result of condition (C) in the definition of φ). Which is in conflict with (4).

Consequently, $P(q_1,q_2,t) > 0, \forall t > 0$.

Similarly, If there exists a $t_0 > 0$ for which $Q(q_1, q_2, t_0) = 1$, then $\varphi(1 - Q(q_1, q_2, t_0)) = \varphi(0) = 0$. This indicates that $1 - Q(q_1, q_2, t_0) = 0$ which implies that $Q(q_1, q_2, t_0) = 1$, Which is in conflict with (5). Consequently, $Q(q_1, q_2, t) < 1, \forall t > 0$.

Following the same scenario, we infer that $\forall n \in N_0$ and $t > 0$

$$0 < P(q_{n-1}, q_n, t) \leq \varphi(P(q_n, q_{n+1}, t)) \leq P(q_n, q_{n+1}, t) < 1$$

$$0 < Q(q_n, q_{n+1}, t) \geq \varphi(Q(q_{n-1}, q_n, t)) \geq Q(q_{n-1}, q_n, t) > 0$$

Which indicates that the sequence $\{P(q_n, q_{n+1}, t)\}$ and $\{Q(q_n, q_{n+1}, t)\}$ are non-decreasing and non-increasing sequences, respectively and bounded.

Therefore, for all $t > 0$, there exist $0 < \delta_1(t) \leq 1$ and $0 < \delta_2(t) \leq 1$ such that

$$\lim_{n\to\infty} P(q_n, q_{n+1}, t) = \delta_1(t)$$

$$\lim_{n\to\infty} Q(q_n, q_{n+1}, t) = \delta_2(t)$$

We will now demonstrate that for all $t > 0$, $\delta_1(t) = 1$ and $\delta_2(t) = 0$.

If there exists $t_0 > 0$ where $\delta_1(t_0) < 1$ then it follows that,

$$0 < P(q_{n-1}, q_n, t_0) \leq \varphi(P(q_n, q_{n+1}, t_0)) \leq P(q_n, q_{n+1}, t_0) \leq \delta_1(t_0) < 1 \quad (6)$$

Thus $0 < \delta_1(t_0) < 1$.

Similarly, If there exists $t_0 > 0$ where $\delta_1(t_0) > 0$ then it follows that,

$$0 < Q(q_n, q_{n+1}, t_0) \geq \varphi(Q(q_{n-1}, q_n, t_0)) \geq Q(q_{n-1}, q_n, t_0) \geq \delta_2(t_0) > 0 \quad (7)$$

Thus $0 < \delta_2(t_0) < 1$.

Given that φ is left-continuous and $\{P(q_n, q_{n+1}, t)\}$ and $\{Q(q_n, q_{n+1}, t)\}$ are sequences of positive numbers that are non-decreasing and non-increasing, respectively, by letting n approach infinity in (6) and (7), we get:

$\varphi(\delta_1(t_0)) = \delta_1(t_0)$, a contradiction ($\delta_1(t) \in (0,1)$).

$1 - \varphi(1 - \delta_2(t_0)) = \delta_2(t_0)$, a contradiction ($\delta_2(t) \in (0,1)$).

Consequently, $\delta_1(t) = 1$ and $\delta_2(t) = 0$, $\forall t > 0$. That is:

$$\lim_{n\to\infty} P(q_n, q_{n+1}, t) = 1 \quad (8)$$

$$\lim_{n\to\infty} Q(q_n, q_{n+1}, t) = 0 \quad (9)$$

Next, we demonstrate that $\{q_n\}$ forms a Cauchy sequence in (I, P, Q, τ, μ) . Conversely, if $\{q_n\}$ is not a Cauchy sequence, then there exist ε within the interval (0, 1) and some $t_0 > 0$ such that for every $k \in N_0$, there are $m(k)$ and $n(k)$ within N_0 such that $k \leq n(k) \leq m(k)$ satisfying:

$$P(q_{m(k)}, q_{n(k)}, t_0) \leq 1 - \varepsilon$$

$$P(q_{m(k)-1}, q_{n(k)}, t_0) > 1 - \varepsilon, \forall k \in N_0$$

Similarly,

$$Q(q_{m(k)}, q_{n(k)}, t_0) \geq \varepsilon$$

$$Q(q_{m(k)-1}, q_{n(k)}, t_0) < \varepsilon, \forall k \in N_0$$

Given that (I, P, Q, τ, μ) is non-Archimedean, it holds true for all $k \in N_0$

$$1 - \varepsilon \geq P(q_{m(k)}, q_{n(k),} t_0)$$

$$\geq P(q_{m(k)}, q_{n(k)-1,} t_0) * P(q_{m(k)-1}, q_{n(k),} t_0)$$

$$> P\left(q_{m(k)}, q_{n(k)-1,} t_0\right) * (1-\varepsilon)$$

Similarly,
$$\varepsilon \le Q(q_{m(k)}, q_{n(k),} t_0)$$
$$\le Q\left(q_{m(k)}, q_{n(k)-1,} t_0\right) \lozenge Q\left(q_{m(k)-1}, q_{n(k),} t_0\right)$$
$$< Q\left(q_{m(k)}, q_{n(k)-1,} t_0\right) \lozenge \varepsilon$$

By letting k approach infinity, and considering the continuity of $*$ and $\diamond$, along with (8) and (9), we can conclude that

$$\lim_{k\to\infty} P\left(q_{m(k)}, q_{n(k)}, t_0\right) = 1-\varepsilon \quad (10)$$

$$\lim_{k\to\infty} Q\left(q_{m(k)}, q_{n(k)}, t_0\right) = \varepsilon \quad (11)$$

Furthermore, given that (I, P, Q, τ, μ) is non-Archimedean, the following holds for all $k \in N_0$

$$P\left(q_{m(k)-1,} q_{n(k)-1}, t_0\right) \ge P\left(q_{m(k)-1,} q_{n(k)}, t_0\right) * P\left(q_{n(k),} q_{n(k)-1}, t_0\right) > (1-\varepsilon) * P\left(q_{n(k)}, q_{n(k)-1}, t_0\right)$$

$$P\left(q_{m(k)}, q_{n(k)}, t_0\right) \ge P\left(q_{m(k)}, q_{m(k)-1}, t_0\right) * P\left(q_{m(k)-1}, q_{n(k)-1}, t_0\right) * P\left(q_{n(k)-1}, q_{n(k)}, t_0\right)$$

$$Q\left(q_{m(k)-1,} q_{n(k)-1}, t_0\right) \le Q\left(q_{m(k)-1,} q_{n(k)}, t_0\right) \lozenge Q\left(q_{n(k),} q_{n(k)-1}, t_0\right) < \varepsilon \lozenge Q\left(q_{n(k)}, q_{n(k)-1}, t_0\right)$$

$$Q\left(q_{m(k)}, q_{n(k)}, t_0\right) \le Q\left(q_{m(k)}, q_{m(k)-1}, t_0\right) \lozenge Q\left(q_{m(k)-1}, q_{n(k)-1}, t_0\right) \lozenge Q\left(q_{n(k)-1}, q_{n(k)}, t_0\right)$$

By allowing k to approach infinity within the above inequalities and applying (8), (9) and (10), (11), we deduce

$$\lim_{k\to\infty} P\left(q_{m(k)-1}, q_{n(k)-1}, t_0\right) = 1-\varepsilon. \quad (12)$$

$$\lim_{k\to\infty} Q\left(q_{m(k)-1}, q_{n(k)-1}, t_0\right) = \varepsilon. \quad (13)$$

Specifically, whenever k is sufficiently large, we have $P(q_{m(k)-1}, q_{n(k)-1}, t_0) > 0$ and $Q(q_{m(k)-1}, q_{n(k)-1}, t_0) < 1$.

Utilizing (2) & (3) and Lemma 2, we obtain the following for all k

$$min\left\{P\left(q_{m(k)-1}, q_{n(k)-1}, t_0\right), max\left\{P\left(hq_{m(k)-1}, q_{m(k)-1}, t_0\right), P\left(q_{n(k)-1}, hq_{n(k)-1}, t_0\right)\right\}\right\} \le \varphi(P(hq_{m(k)-1}, hq_{n(k)-1}, t_0))$$

$$max\left\{Q\left(q_{m(k)-1}, q_{n(k)-1}, t_0\right), min\left\{Q\left(hq_{m(k)-1}, q_{m(k)-1}, t_0\right), Q\left(q_{n(k)-1}, hq_{n(k)-1}, t_0\right)\right\}\right\} \ge \varphi(Q(hq_{m(k)-1}, hq_{n(k)-1}, t_0))$$

Hence,

$$min\left\{P\left(q_{m(k)-1}, q_{n(k)-1}, t_0\right), max\left\{P\left(q_{m(k)}, q_{m(k)-1}, t_0\right), P\left(q_{n(k)-1}, q_{n(k)}, t_0\right)\right\}\right\} \le \varphi(P(q_{m(k)}, q_{n(k)}, t_0))$$

$$max\left\{Q\left(q_{m(k)-1}, q_{n(k)-1}, t_0\right), min\left\{Q\left(q_{m(k)}, q_{m(k)-1}, t_0\right), Q\left(q_{n(k)-1}, q_{n(k)}, t_0\right)\right\}\right\} \ge \varphi(Q(q_{m(k)}, q_{n(k)}, t_0))$$

By allowing k to approach infinity and applying (8) – (13) along with the left-continuity of φ, we conclude that:

$1-\varepsilon < min\{1-\varepsilon, max\{1,1\}\} \le \varphi(1-\varepsilon) \Rightarrow 1-\varepsilon \le \varphi(1-\varepsilon) < 1-\varepsilon,$ a contradiction.

$\varepsilon > max\{\varepsilon, min\{0,0\}\} \ge \varphi(\varepsilon) \Rightarrow \varepsilon \ge \varphi(\varepsilon) > \varepsilon,$ a contradiction.

Therefore, $\{q_n\}$ must form a Cauchy sequence in (I, P, Q, τ, μ). Given that $\{q_n\}$ is an $\mathfrak{R}$-Cauchy sequence and (I, P, Q, τ, μ) is $\mathfrak{R}$-complete, there exists an $q \in I$ such that q_n converges to q.

If f is continuous, then by taking the limit as n approaches infinity on both sides of $q_{n+1} = hq_n,\ n \in N_0$, we obtain $q = hq$.

Otherwise, if $\mathfrak{R}$ is P-self-closed and Q-self-closed, there exists a subsequence $\{q_{n(k)}\} \subseteq \{q_n\}$ such that $\{q_{n(k)}\}\mathfrak{R}q$ for all $k \in N_{0.}$

We state that $q = h(q)$. Given that the $\lim_{k\to\infty} q_{n(k)} = q$, we have

$$\lim_{k\to\infty} P(q_n, q, t) = 1, \forall t > 0$$

$$\lim_{k\to\infty} Q(q_n, q, t) = 0, \forall t > 0$$

Thus, for all $t > 0$, when k is sufficiently large, we have $P(q_n, q, t) > 0$ and $Q(q_n, q, t) < 1$. Given that $q_{n(k)}\Re q$, according to condition (2) & (3), we find

$$min\left\{P\left(q_{n(k)}, q, t\right), max\left\{P\left(hq_{n(k)}, q_{n(k)}, t\right), P(q, hq, t)\right\}\right\} \leq \varphi\left(P\left(hq_{n(k)}, hq, t\right)\right).$$

$$max\left\{Q\left(q_{n(k)}, q, t\right), min\left\{Q\left(hq_{n(k)}, q_{n(k)}, t\right), Q(q, hq, t)\right\}\right\} \geq \varphi\left(Q\left(hq_{n(k)}, hq, t\right)\right).$$

Thus,

$$min\left\{P\left(q_{n(k)}, q, t\right), max\left\{P\left(q_{n(k)+1}, q_{n(k)}, t\right), P(q, hq, t)\right\}\right\} \leq \varphi\left(P\left(q_{n(k)+1}, hq, t\right)\right).$$

$$max\left\{Q\left(q_{n(k)}, q, t\right), min\left\{Q\left(q_{n(k)+1}, q_{n(k)}, t\right), Q(q, hq, t)\right\}\right\} \geq \varphi\left(Q\left(q_{n(k)+1}, hq, t\right)\right).$$

By allowing $k \to \infty$ and utilizing equations (8) and (9), we observe that $\lim_{k\to\infty} P\left(q_{n(k)}, q, t\right) = 1$ and $\lim_{k\to\infty} Q\left(q_{n(k)}, q, t\right) = 0$ leading to the conclusion

$$1 = min\left\{1, max\{1, P(q, hq, t)\}\right\} \leq \lim_{k\to\infty} \varphi(P(q_{n(k)+1}, hq, t))$$

$$0 = max\left\{0, min\{1, Q(q, hq, t)\}\right\} \geq \lim_{k\to\infty} \varphi(Q(q_{n(k)+1}, hq, t))$$

As a result, this indicates that

$$\lim_{k\to\infty} \varphi\left(P\left(q_{n(k)+1}, hq, t\right)\right) = 1$$

$$\lim_{k\to\infty} \varphi\left(Q\left(q_{n(k)+1}, hq, t\right)\right) = 0$$

Consequently, based on Remark 4 (iii) and the continuity of P and Q, we deduce

$$\lim_{k\to\infty} P(q_n, q, t) = 1, \forall t > 0$$

$$\lim_{k\to\infty} Q(q_n, q, t) = 0, \forall t > 0$$

Hence, as k approaches infinity, $\lim_{k\to\infty} q_{n(k)+1} = hq$. The unique limit implies that $hq = q$. This concludes the proof.

Following this, we present the subsequent uniqueness theorem.

Theorem2: Building on the hypotheses of Theorem 1, if the following condition is satisfied:

iv. For all $q, p \in Fix(h)$, there exists $r \in I$ such that $q\Re r$ and $p\Re r$, with
$P(q, r, t) > 0, Q(q, r, t) < 1, P(p, r, t) > 0, Q(p, r, t) < 1, \forall t > 0.$
Then, the fixed point of h is unique.

Proof: Considering Theorem 1, $Fix(h)$ is not empty. Suppose q and p are elements of $Fix(h)$.

According to condition (iv), there exists an element r in I such that $q\Re r$ and $p\Re r$, with

$P(q, r, t) > 0, Q(q, r, t) < 1, P(p, r, t) > 0, Q(p, r, t) < 1, \forall t > 0.$

Let $r_0 = r$ and $r_{n+1} = hr_n, \forall n \geq 0$. We claim that $q = p$. Given that $q\Re r_0$, and $P(q, r_0, t) > 0, Q(q, r_0, t) < 1\ \forall t > 0$, it follows from equation (2) & (3) that

$$min\left\{P(q, r_0, t), max\{P(hq, q, t), P(r_0, hr_0, t)\}\right\} \leq \varphi(P(hq, hq_0, t))$$

$$\Rightarrow \quad min\left\{P(q, r_0, t), max\{P(q, q, t), P(r_0, r_1, t)\}\right\} \leq \varphi(P(q, r_1, t))$$

$$\Rightarrow \quad min\left\{P(q, r_0, t), max\{1, P(r_0, r_1, t)\}\right\} \leq \varphi(P(q, r_1, t))$$

$$\Rightarrow \quad min\{P(q, r_0, t), 1\} \leq \varphi(P(q, r_1, t))$$

$$\Longrightarrow \quad 0 < P(q, r_0, t) \leq \varphi\big(P(q, r_1, t)\big) \leq P(q, r_1, t)$$

And $\quad max\{Q(q, r_0, t), min\{Q(hq, q, t), Q(r_0, hr_0, t)\}\} \geq \varphi(P(hq, hq_0, t))$

$$\Longrightarrow \quad max\{Q(q, r_0, t), min\{Q(q, q, t), Q(r_0, r_1, t)\}\} \geq \varphi(Q(q, r_1, t))$$

$$\Longrightarrow \quad max\{Q(q, r_0, t), min\{1, Q(r_0, r_1, t)\}\} \geq \varphi(Q(q, r_1, t))$$

$$\Longrightarrow \quad max\{Q(q, r_0, t), 1\} \geq \varphi(Q(q, r_1, t))$$

$$\Longrightarrow \quad 0 < Q(q, r_0, t) \geq \varphi\big(Q(q, r_1, t)\big) \geq Q(q, r_1, t)$$

Through induction, we establish that $P(q, r_n, t) > 0$ and $Q(q, r_n, t) < 1 \; \forall n \in N_0$ and $t > 0$. Given that $\Re$ is h-closed, we deduce (by induction) that $q\Re r_n, \forall n \in N_0$. Hence,

$$min\{P(q, r_n, t), max\{P(hq, q, t), P(r_n, hr_n, t)\}\} \leq \varphi(P(hq, hr_n, t))$$

$$\Longrightarrow \quad min\{P(q, r_n, t), max\{P(q, q, t), P(r_n, r_{n+1}, t)\}\} \leq \varphi\big(P(q, r_{n+1}, t)\big) \qquad (14)$$

$$\Longrightarrow \quad min\{P(q, r_n, t), max\{1, P(r_n, hr_{n+1}, t)\}\} \leq \varphi(P(q, r_{n+1}, t))$$

$$\Longrightarrow \quad min\{P(q, r_n, t), 1\} \leq \varphi(P(q, r_{n+1}, t))$$

$$\Longrightarrow \quad 0 < P(q, r_n, t) \leq \varphi(P(q, r_{n+1}, t) \leq P(q, r_{n+1}, t)$$

and

$$max\{Q(q, r_n, t), min\{Q(hq, q, t), Q(r_n, hr_n, t)\}\} \geq \varphi(Q(hq, hr_n, t))$$

$$\Longrightarrow \quad max\{Q(q, r_n, t), min\{Q(q, q, t), Q(r_n, r_{n+1}, t)\}\} \geq \varphi\big(Q(q, r_{n+1}, t)\big) \qquad (15)$$

$$\Longrightarrow \quad max\{Q(q, r_n, t), min\{0, Q(r_n, hr_{n+1}, t)\}\} \geq \varphi(Q(q, r_{n+1}, t))$$

$$\Longrightarrow \quad min\{Q(q, r_n, t), 0\} \geq \varphi(Q(q, r_{n+1}, t))$$

$$\Longrightarrow \quad 0 < Q(q, r_n, t) \geq \varphi(Q(q, r_{n+1}, t) \geq Q(q, r_{n+1}, t)$$

Consequently, $\{P(q, r_n, t)\}$ is non-decreasing and capped at an upper limit, while $\{Q(q, r_n, t)\}$ is non-increasing and limited from below. Therefore, for all $t > 0$, there exist values $0 < \gamma_1(t) \leq 1$ and $0 < \gamma_2(t) \leq 1$ such that

$$\lim_{n \to \infty} P(q, r_n, t) = \gamma_1(t)$$

$$\lim_{n \to \infty} Q(q, r_n, t) = \gamma_2(t)$$

By allowing $n \to \infty$ in equation (14) & (15), and considering that ψ is left-continuous, it follows that:

$$\varphi\big(\gamma_1(t)\big) = \gamma_1(t)$$

$$1 - \varphi\big(\gamma_2(t)\big) = \gamma_2(t)$$

Accordingly, considering Remark 4, we conclude that

$$\gamma_1(t) = 1, \forall t > 0$$

$$\gamma_2(t) = 0, \forall t > 0$$

Thus,

$$\lim_{n \to \infty} P(q, r_n, t) = 1 \, , \forall t > 0$$

$$\lim_{n \to \infty} Q(q, r_n, t) = 0, \forall t > 0$$

In a similar way, we can demonstrate that

$$\lim_{n\to\infty} P(p, r_n, t) = 1\,, \forall t > 0$$

$$\lim_{n\to\infty} Q(p, r_n, t) = 0, \forall t > 0$$

Since (I, P, Q, τ, μ) is non-Archimedean, it follows that (for all $n \in N_0$)

$$P(q, p, t) \geq P(q, r_n, t) * P(r_n, p, t)$$

$$Q(q, p, t) \leq Q(q, r_n, t) \lozenge Q(r_n, p, t)$$

As n approaches infinity and considering the continuity of τ and μ, we deduce that:

$$P(q, p, t) \geq \tau(1,1) = 1$$

$$Q(q, p, t) \leq \mu(0,0) = 0$$

Hence,

$$P(q, p, t) = 1$$

$$Q(q, p, t) = 0$$

Thus, we arrive at the conclusion that $q = p$, fulfilling our requirements.

Corollary 1: Consider a space (I, P, Q, τ, μ) which is an $\mathfrak{R}$-complete non-Archimedean intuitionistic fuzzy metric space, with $\mathfrak{R}$ being a binary relation and a mapping $h: I \longrightarrow I$. Suppose there exists $k \in (0,1)$ such that for any $q, p \in I$ and all $t > 0$ where $q\mathfrak{R}p$, if $P(q, p, t) > 0$ and $P(q, p, t) > 0 \Rightarrow min\{P(q, p, t), max\{P(hq, q, t), P(p, hp, t)\}\} \leq kP(hq, hp, t)$

$Q(q, p, t) < 1 \Rightarrow max\{Q(q, p, t), min\{Q(hq, q, t), Q(p, hp, t)\}\} \geq kQ(hq, hp, t)$

Moreover,

i. there exists $q_0 \in I$ such that $q_0\mathfrak{R}hq_0$, where $P(q_0, hq_0, t) > 0$ and $Q(q_0, hq_0, t) < 1$ for all $t > 0$.
ii. $\mathfrak{R}$ exhibits transitivity and is closed under the function h.
iii. One of the following conditions is true:
   a. The function h exhibits continuity or
   b. Relation $\mathfrak{R}$ is P- self closed and Q-self closed.

Consequently, h possesses a fixed point in I. Moreover, this is true provided that the following condition is met.

iv. For every q and p in $Fix(h)$, there exists a r in I such that $q\mathfrak{R}r$ and $p\mathfrak{R}r$, with $P(q, r, t) > 0, Q(q, r, t) < 1$ and $P(p, r, t) > 0, Q(p, r, t) < 1$ for all $t > 0$.
   Then the fixed point is unique.

In the following sections, we demonstrate that Theorems 1 and 2 can be established within the framework of $\mathfrak{R}$-complete non-Archimedean intuitionistic fuzzy metric spaces (based on definition of fuzzy metric space defined by George and Veeramani). Subsequently, we introduce the concept of GV-intuitionistic fuzzy $\mathfrak{R} - \varphi$-contractive.

Definition 18: Consider (I, P, Q, τ, μ) as a non-Archimedean intuitionistic fuzzy metric space. Let $\mathfrak{R}$ be a binary relation on I and $h: I \to I$. We define h as a intuitionistic fuzzy $\mathfrak{R} - \varphi$ contractive mapping if there exist $\varphi \in \Omega$ such that $\forall\, q, p \in I, t > 0$ with $q\mathfrak{R}p$.

$$min\{P(q, p, t), max\{P(hq, q, t), P(p, hp, t)\}\} \leq \varphi\big(P(hq, hp, t)\big). \qquad (16)$$

$$max\{Q(q, p, t), min\{Q(hq, q, t), Q(p, hp, t)\}\} \geq \varphi\big(Q(hq, hp, t)\big). \qquad (17)$$

Theorem 3: Consider (I, P, Q, τ, μ) to be a non-Archimedean intuitionistic fuzzy metric space with a binary relation $\mathfrak{R}$ and a mapping $h: I \to I$. Suppose I is $\mathfrak{R}$- complete and h is intuitionistic fuzzy $\mathfrak{R} - \varphi -$ contractive mapping such that:

i. There is $q_0 \in I$ such that $q_0\mathfrak{R}hq_0$;
ii. Relation $\mathfrak{R}$ is transitive and closed under h.

iii. One of the following conditions is true:
  a. The function h is continuous or
  b. Relation $\Re$ is P- self closed and Q-self closed.

  Consequently, h possesses a fixed point within I.

Proof: We can find $q_0 \in I$ from (i) such that $q_0 \Re h q_0$. Let a sequence $\{q_0\}$ in I where $hq_n = q_{n+1}, \ \forall n \in \mathbb{N}_0$. If $q_n = q_{n+1}$ for some $n \in N_0$, then q_n is fixed point of h. Suppose $q_n \neq q_{n+1}$ for all $n \in \mathbb{N}_0$. Given that $q_0 \Re q_1$ and considering equation (16) & (17), we derive the following:

$$min\{P(q_0, q_1, t), P(hq_0, q_0, t), P(q_1, hq_1, t)\} \leq \varphi(P(hq_0, hq_1, t))$$

$$\Rightarrow \quad min\{P(q_0, q_1, t), P(q_1, q_0, t), P(q_1, q_2, t)\} \leq \varphi\big(P(q_1, q_2, t)\big) \quad (18)$$

Similarly,

$$max\{Q(q_0, q_1, t), Q(hq_0, q_0, t), Q(q_1, hq_1, t)\} \geq \varphi(Q(hq_0, hq_1, t))$$

$$\Rightarrow \quad max\{Q(q_0, q_1, t), Q(q_1, q_0, t), Q(q_1, q_2, t)\} \geq \varphi\big(Q(q_1, q_2, t)\big) \quad (19)$$

If $\quad min\{P(q_0, q_1, t), P(q_1, q_2, t)\} = P(q_1, q_2, t) \Rightarrow \varphi\big(P(q_1, q_2, t)\big) = P(q_1, q_2, t),$

then we have, $P(q_1, q_2, t) = 1$, leading to a contradiction.

and $\quad max\{Q(q_0, q_1, t), Q(q_1, q_2, t)\} = Q(q_1, q_2, t) \Rightarrow \varphi\big(Q(q_1, q_2, t)\big) = Q(q_1, q_2, t),$

then we have, $Q(q_1, q_2, t) = 0$, leading to a contradiction.

Therefore,

$$0 < P(q_0, q_1, t) \leq \varphi(P(q_1, q_2, t)) \leq P(q_1, q_2, t)$$

$$0 < Q(q_0, q_1, t) \geq \varphi(Q(q_1, q_2, t)) \geq Q(q_1, q_2, t)$$

By repeating this procedure, we conclude that

$$0 < P(q_{n-1}, q_n, t) \leq \varphi\big(P(q_n, q_{n+1}, t)\big) \leq P(q_n, q_{n+1}, t),$$

$$0 < Q(q_{n-1}, q_n, t) \geq \varphi\big(Q(q_n, q_{n+1}, t)\big) \geq Q(q_n, q_{n+1}, t)$$

For every $n \in \mathbb{N}_0$, in the proof of Theorem 1, we have

$$\begin{aligned} \lim_{n\to\infty} P(q_n, q_{n+1}, t) &= 1 \\ \lim_{n\to\infty} Q(q_n, q_{n+1}, t) &= 0 \end{aligned} \quad (20)$$

Next, we demonstrate that $\{q_n\}$ is a Cauchy sequence in (I, P, Q, τ, μ). Conversely, if $\{q_n\}$ is not a Cauchy sequence, then according to the proof of Theorem 1, we obtain

$$\begin{aligned} \lim_{k\to\infty} P\big(q_{m(k)}, q_{n(k)}, t_0\big) &= 1 - \varepsilon \\ \lim_{k\to\infty} Q\big(q_{m(k)}, q_{n(k)}, t_0\big) &= \varepsilon \end{aligned} \quad (21)$$

$$\begin{aligned} \lim_{k\to\infty} P\big(q_{m(k)-1}, q_{n(k)-1}, t_0\big) &= 1 - \varepsilon \\ \lim_{k\to\infty} Q\big(q_{m(k)-1}, q_{n(k)-1}, t_0\big) &= \varepsilon \end{aligned} \quad (22)$$

By applying the contractive conditions (16) & (17) and Lemma 2, we obtain for every k

$$min\{P\big(q_{m(k)-1}, q_{n(k)-1}, t_0\big), P\big(hq_{m(k)-1}, q_{m(k)-1}, t_0\big), P\big(q_{n(k)-1}, hq_{n(k)-1}, t_0\big)\} \leq \varphi(P\big(hq_{m(k)-1}, hq_{n(k)-1}, t_0\big)$$

and

$$max\{Q\big(q_{m(k)-1}, q_{n(k)-1}, t_0\big), Q\big(hq_{m(k)-1}, q_{m(k)-1}, t_0\big), Q\big(q_{n(k)-1}, hq_{n(k)-1}, t_0\big)\} \leq \varphi(Q\big(hq_{m(k)-1}, hq_{n(k)-1}, t_0\big)$$

$$min\{P(q_{m(k)-1}, q_{n(k)-1}, t_0), P(q_{m(k)}, q_{m(k)-1}, t_0), P(q_{n(k)-1}, q_{n(k)}, t_0) \leq \varphi(P(q_{m(k)}, q_{n(k)}, t_0))\}$$
$$max\{Q(q_{m(k)-1}, q_{n(k)-1}, t_0), Q(q_{m(k)}, q_{m(k)-1}, t_0), Q(q_{n(k)-1}, q_{n(k)}, t_0) \leq \varphi(Q(q_{m(k)}, q_{n(k)}, t_0))\}$$

As $k \to \infty$, and by utilizing equations (20)-(22) along with the left-continuity of φ, it can be determined that

$$1 - \varepsilon < \min\{1 - \varepsilon, 1,1\} \leq \varphi(1 - \varepsilon) \leq 1$$

and

$$\varepsilon > max\{\varepsilon, 0,0\} \geq \varphi(\varepsilon) \geq 0$$

This implies that $\varphi(1 - \varepsilon) = 1 - \varepsilon$ and $\varphi(\varepsilon) = \varepsilon$, which is a contradiction.

Therefore, the sequence $\{q_n\}$ in (I, P, Q, τ, μ) must be a Cauchy sequence. Given that (I, P, Q, τ, μ) is $\mathfrak{R}$-complete, there exists an element q in I such that $q_n \to q$. According to condition (a), if the function h is continuous, we can deduce from the proof of Theorem 1 that

$$\lim_{n\to\infty} P(q_n, hq, t) = 1$$

and

$$\lim_{n\to\infty} Q(q_n, hq, t) = 0$$

which implies that $P(q, hq, t) = 1$ and $Q(q, hq, t) = 0$, and hence $q = hq$.

From condition (b) if $\mathfrak{R}$ is (P, Q)–self-closed, then there exists a subsequence $\{q_{n(k)}\} \subseteq \{q_n\}$ such that $\lim_{k\to\infty} q_{n(k)} = q$ and $q_{n(k)}\mathfrak{R}q$ for all $k \in \mathbb{N}_0$.

Assume $q \neq h(q)$. From condition (16) & (17), we can deduce the following

$$min\{P(q_{n(k)}, q, t), P(hq_{n(k)}, q_{n(k)}, t), P(q, hq, t)\} \leq \varphi(P(hq_{n(k)}, hq, t))$$

and

$$max\{Q(q_{n(k)}, q, t), Q(hq_{n(k)}, q_{n(k)}, t), Q(q, hq, t)\} \geq \varphi(Q(hq_{n(k)}, hq, t))$$

Thus,

$$min\{P(q_{n(k)}, q, t), P(q_{n(k)+1}, q_{n(k)}, t), P(q, hq, t)\} \leq \varphi(P(q_{n(k)+1}, hq, t))$$
$$max\{Q(q_{n(k)}, q, t), Q(q_{n(k)+1}, q_{n(k)}, t), Q(q, hq, t)\} \geq \varphi(Q(q_{n(k)+1}, hq, t))$$

By allowing $k \to \infty$ and applying equation (20), we get

$$\lim_{k\to\infty} P(q_{n(k)}, q, t) = 1$$
$$\lim_{k\to\infty} Q(q_{n(k)}, q, t) = 0$$

$$P(q, hq, t) = min\{1,1, P(q, hq, t)\} \leq \lim_{k\to\infty} \varphi(P(q_{n(k)+1}, hq, t))$$
$$Q(q, hq, t) = max\{0,0, Q(q, hq, t)\} \geq \lim_{k\to\infty} \varphi(Q(q_{n(k)+1}, hq, t))$$

Given the continuity of φ, it follows that

$$P(q, hq, t) \leq \varphi(P(q, hq, t)) < P(q, hq, t)$$
$$Q(q, hq, t) \geq \varphi(Q(q, hq, t)) > Q(q, hq, t)$$

Therefore, based on Remark 4 (iii), we obtain $P(q, hq, t) = 1$ and $Q(q, hq, t) = 0$, as needed. Thus, it follows that $hq = q$.

Following this, we present the subsequent uniqueness theorem.

Theorem 4: Assuming the same conditions as Theorem 3, an additional requirement is:

iv. For every $q, p \in Fix(h)$, there exists a $r \in I$ such that $q\mathfrak{R}r, p\mathfrak{R}r$ and $r\mathfrak{R}hr$. With this condition, the fixed point of h is unique.

Proof: Based on Theorem 3, $Fix(h) \neq \phi$. Let $q, p \in Fix(h)$. According to condition (iv), there exists $r \in I$ such that $q\Re r$ and $p\Re r$. Define $r_{n+1} = h(r_n)$ for all $n \geq 0$ with $r_0 = r$. Since $r\Re hr$, then following the proof of Theorem 3, we have

$$\begin{aligned} \lim_{n\to\infty} P(r_n, r_{n+1}, t) = 1 \\ \lim_{n\to\infty} Q(r_n, r_{n+1}, t) = 0 \end{aligned} \tag{23}$$

This indicates that $\{r_n\}$ forms a Cauchy sequence within the space (I, P, Q, τ, μ).

We assert that $q = p$. Since $q\Re r_0$ and $\Re$ is h-closed, it follows by induction that $q\Re r_n$ for all $n \in \mathbb{N}_0$. Thus, utilizing equation (16) & (17), we obtain

$$min\{P(q, r_n, t), P(hq, q, t), P(r_n, hr_n, t)\} \leq \varphi(P(hq, hr_n, t))$$

$$\Longrightarrow \quad min\{P(q, r_n, t), P(q, q, t), P(r_n, r_{n+1}, t)\} \leq \varphi(P(q, r_{n+1}, t))$$

$$\Longrightarrow \quad min\{P(q, r_n, t), P(r_n, r_{n+1}, t)\} \leq \varphi(P(q, r_{n+1}, t))$$

and

$$max\{Q(q, r_n, t), Q(hq, q, t), Q(r_n, hr_n, t)\} \geq \varphi(Q(hq, hr_n, t))$$

$$\Longrightarrow \quad max\{Q(q, r_n, t), Q(q, q, t), Q(r_n, r_{n+1}, t)\} \geq \varphi(Q(q, r_{n+1}, t))$$

$$\Longrightarrow \quad max\{Q(q, r_n, t), Q(r_n, r_{n+1}, t)\} \geq \varphi(Q(q, r_{n+1}, t))$$

Case I: When $min\{P(q, r_n, t), P(r_n, r_{n+1}, t)\} = P(q, r_n, t)$ and $max\{Q(q, r_n, t), Q(r_n, r_{n+1}, t)\} = Q(q, r_n, t)$ for all $n \geq n_0$, it follows that:

$$\begin{gathered} P(q, r_n, t) \leq \varphi(P(q, r_{n+1}, t)) \leq P(q, r_{n+1}, t) \\ Q(q, r_n, t) \geq \varphi(Q(q, r_{n+1}, t)) \geq Q(q, r_{n+1}, t) \end{gathered}$$

Thus, $\{P(q, r_n, t)\}$ is non-decreasing and bounded above, and $\{Q(q, r_n, t)\}$ is non-increasing and bounded below. Therefore, as demonstrated in Theorem 2

$$\begin{gathered} \lim_{n\to\infty} P(q, r_n, t) = 1 \\ \lim_{n\to\infty} Q(q, r_n, t) = 0 \end{gathered}$$

$$\Longrightarrow \quad \lim_{n\to\infty} r_n = p$$

Case II: When $min\{P(q, r_n, t), P(r_n, r_{n+1}, t)\} = P(q, r_{n+1}, t)$ and $max\{Q(q, r_n, t), Q(r_n, r_{n+1}, t)\}$

$= Q(q, r_{n+1}, t)$ for all $n \geq n_0$, it follows that:

$$\begin{gathered} P(r_n, r_{n+1}, t) \leq \varphi(P(q, r_{n+1}, t)) \\ Q(r_n, r_{n+1}, t) \geq \varphi(Q(q, r_{n+1}, t)) \end{gathered}$$

By letting $n \longrightarrow \infty$ and applying equation (23), we obtain

$$1 \leq \lim_{n\to\infty} \varphi(P(q, r_{n+1}, t))$$

Since φ is continuous, we get

$$\Longrightarrow \quad 1 = \lim_{n\to\infty} \varphi(P(q, r_{n+1}, t))$$

$$\Longrightarrow \quad 1 = \lim_{n\to\infty} P(q, r_{n+1}, t)$$

$$\Longrightarrow \quad \lim_{n\to\infty} r_{n+1} = q$$

and

$$0 \leq \lim_{n\to\infty} \varphi(Q(q, r_{n+1}, t))$$

Since φ is continuous, we get

$$\Rightarrow \qquad 0 = \lim_{n\to\infty} \varphi(Q(q, r_{n+1}, t))$$

$$\Rightarrow \qquad 0 = \lim_{n\to\infty} Q(q, r_{n+1}, t)$$

$$\Rightarrow \qquad \lim_{n\to\infty} r_{n+1} = q$$

Consequently, by considering both cases, we determine that as

$$\lim_{n\to\infty} r_n = q \tag{24}$$

In a similar manner, it can be demonstrated that as

$$\lim_{n\to\infty} r_n = p \tag{25}$$

Given that (I, P, Q, τ, μ) is a Hausdorff space, we derive from equations (24) and (25) that $q = p$. This concludes the proof.

By substituting $\varphi(t) = kt$, where $k \in (0,1)$, in Theorems 3 and 4, we derive the following corollary.

Corollary 2. Consider (I, P, Q, τ, μ) as an $\Re$–complete non-Archimedean intuitionistic fuzzy metric space with a binary relation $\Re$. Let $h: I \longrightarrow I$ be a function such that there exists $k \in (0,1)$ and for all $q, p \in I$, with $q\Re p$:

$$min\{P(q,p,t), P(hq,q,t), P(p,hp,t)\} \leq kP(hq,hp,t)$$
$$max\{Q(q,p,t), Q(hq,q,t), Q(p,hp,t)\} \geq kQ(hq,hp,t)$$

Additionally, the following conditions are holds:

(i) there exists a point $q_0 \in I$ such that $q_0 \Re hq_0$;
(ii) the relation $\Re$ is both transitive and h-closed;
(iii) one of the two conditions is satisfied:
(a) the function h is continuous, or
(b) the relation $\Re$ is (P, Q)-self-closed.

Under these conditions, the function h has at least one fixed point in the set I.

Furthermore, if the following condition is satisfied:

(iv) for every $q, p \in Fix(h)$, there is a $r \in I$ such that $q\Re r, p\Re r$ and $r\Re hr$.

Then the fixed point is unique.

4. Application to Non-Linear Fractional Differential Equations

In this part, we utilize our primary findings to investigate the existence of solutions to boundary value problems for fractional differential equations that incorporate the Caputo fractional derivative.

Consider $I = C([0,1], \mathbb{R})$, the Banach space with all continuous functions from [0,1] to $\mathbb{R}$, equipped with the norm

$$\|q\|_\infty = \sup_{t\in[0,1]} |x(t)|$$

Define $P: I^2 \times (0, \infty) \to [0,1]$ and $Q: I^2 \times (0, \infty) \to [0,1]$ for all $q, p \in I$, by

$$P(q,p,t) = e^{\frac{-\|q-p\|_\infty}{t}} \qquad \forall t \in (0,1)$$

$$Q(q,p,t) = 1 - e^{\frac{-\|q-p\|_\infty}{t}} \qquad \forall t \in (0,1)$$

It is widely recognized that (I, P, Q, τ, μ) represents a complete non-Archimedean intuitionistic fuzzy metric space, where $\tau(a, b) = a.b$ and $\mu(a, b) = min\{a, b\}$ for all $a, b \in [0,1]$ (refer to sources ([4],[20]). Let us define a binary relation $\Re$ on I by

$$q\Re p \Leftrightarrow q(t) \leq p(t) \text{ for all } q, p \in I, t \in [0,1]$$

Given that (I, P, Q, τ, μ) represents a complete non-Archimedean intuitionistic fuzzy metric space where $a * b = a.b$ and $a \lozenge b = min\{a, b\}$ for all $a, b \in [0,1]$, it follows that (I, P, Q, τ, μ) is also an $\Re$-complete non-Archimedean intuitionistic fuzzy metric space with the same operation $\tau(a, b) = a.b$ and $\mu(a, b) = min\{a, b\}$ for all $a, b \in [0,1]$. Moreover, it is apparent that $\Re$ shows transitivity.

Let's review the fundamental concepts that will be required later.

Definition 19([10]): For a function u given on the interval $[a, b]$ the Caputo fractional derivative of function û order $\beta > 0$ is defined by

$$\left({}^{c}D_{a^+}^{\beta}\right)\hat{u}(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t (t - s)^{n-\beta-1} \hat{u}^{(n)}(n) ds, \quad (n - 1 \le \beta < n, n = [\beta] + 1), \tag{26}$$

where $[\beta]$ denotes the integer part of the positive real number β and Γ is a gamma function.

Consider the boundary value problem for fractional order differential equation given by:

$$\begin{cases} {}^{c}D_{0^+}^{\beta}\big(q(t)\big) = f\big(t, q(t)\big), \ (t \in [0,1], 2 < \beta \le 3) \\ \quad q(0) = c_0, \ \ q'(0) = c_0^*, \ q''(1) = c_1 \end{cases} \tag{27}$$

where ${}^{c}D_{0^+}^{\beta}$ denotes the Caputo fractional derivative of order β, $f: [0,1] \longrightarrow R$ is a continuous function and c_0, c_0^*, c_1 are real constants.

Definition 20([1]): A function $q \in c^3([0,1], \mathbb{R})$ whose β-derivative existing on $[0,1]$ is considered a solution of equation (18) if q satisfies ${}^{c}D_{0^+}^{\beta}\big(q(t)\big) = f\big(t, q(t)\big)$ on [0, 1] along the conditions $q(0) = c_0, \ q'(0) = c_0^*, \ q''(1) = c_1$.

The subsequent lemma is essential for the result.

Lemma 3([1]): Consider β within the range $2 < \beta \le 3$, and let û, a continuous function, û: $[0,1] \longrightarrow \mathbb{R}$. A function x is a solution to the fractional integral equation

$$q(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t + s)^{\beta} \hat{u}(s)\, ds - \frac{t^2}{2\Gamma(\beta - 2)} \int_0^1 (1 - s)^{\beta-3} \hat{u}(s) ds + c_0 + c_0^* t + \frac{c_1}{2} t^2$$

if and only if q is a solution of the fractional boundary value problems

$${}^{c}D_{0^+}^{\beta}\big(q(t)\big) = \hat{u}(t)$$

$$q(0) = c_0, \ \ q'(0) = c_0^*, \ q''(1) = c_1$$

$$q''(1) = 2c_2$$

Where, $\quad q''(1) = 2c_2 + \frac{1}{\Gamma(\beta+2)} \int_0^1 (1 - s)^{\beta-3} \hat{u}(s) ds = c_1, c_i, c_0^* \in \mathbb{R}, i = 0,1,2.$

In this section, we present and demonstrate our main result.

Theorem 5: Assume that

(i)

$$|f(t, q(t) - f(t, p(t))| \le \lambda |q(t) - p(t)|, \text{ where}$$

$$0 < \frac{1}{k} = \lambda \left(\frac{1}{\Gamma(\beta+1)} + \frac{1}{2\Gamma(\beta-1)}\right) < 1 \tag{28}$$

(ii) there exist $q_0 \in I$ such that

$$q_0(t) \le \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} h\big(s, q_0(s)\big) ds - \frac{t}{2\Gamma(\beta - 2)} \int_0^1 (1 - s)^{\beta-3} f\big(s, q_0(s)\big) ds + c_0 + c_0^* t + \frac{c_1}{2} t^2$$

(iii) The function f non-decreasing with respect to its second variable.

Consequently, Equation (27) possesses a unique solution within the set I.

Proof: Let $\mathcal{H}: I \longrightarrow I$ be defined as follows

$$\mathcal{H}q(\mathfrak{t}) = \frac{1}{\Gamma(\beta)}\int_0^t (\mathfrak{t}-s)^{\beta-1} f\big(s,q(s)\big)ds - \frac{t}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3} f(s,q(s))ds + c_0 + c_0^*\mathfrak{t} + \frac{c_1}{2}\mathfrak{t}^2$$

where, $c_1 = 2c_2 + \frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\beta-3} f(s,q(s))ds,\quad c_i, c_0^*\mathfrak{t} \in \mathbb{R}, (i = 0,1,2)$ are constant.

First, we demonstrate the continuity of $\mathcal{H}$. Consider a sequence $\{q_n\}$ where $\lim_{n\to\infty} q_n = q$ in I. For each $\mathfrak{t}$ within the interval [0,1], the following holds

$$|\mathcal{H}q_n(\mathfrak{t}) - \mathcal{H}q(\mathfrak{t})| \le \frac{1}{\Gamma(\beta)}\int_0^t (\mathfrak{t}-s)^{\beta-1}\big|f\big(s,q_n(s)\big) - f(s,q(s))\big|ds + \frac{1}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}|f(s,q_n(s)) - f(s,q(s))ds|$$

Given that f is a continuous function, it follows that

$$\lim_{n\to\infty}\big\|f\big(s,q_n(s)\big) - f(s,q(s))\big\|_\infty = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \|\mathcal{H}q_n - \mathcal{H}q\|_\infty = 0$$

$$\Leftrightarrow \lim_{n\to\infty} e^{\frac{-\|\mathcal{H}q_n - \mathcal{H}q\|_\infty}{t}} = 1$$

$$\Leftrightarrow \lim_{n\to\infty} M(\mathcal{H}q_n, \mathcal{H}q, t) = 1$$

$$\Leftrightarrow \lim_{n\to\infty} \mathcal{H}q_n = \mathcal{H}q$$

Therefore, $\mathcal{H}$ is continuous.

Clearly, the fixed points of the operator correspond to the solutions of Equation (27). To demonstrate that $\mathcal{H}$ has a fixed point, we will apply Theorem 3.

Thus, we demonstrate that $\mathcal{H}$ is a GV-intuitionistic fuzzy $\Re - \varphi -$ contractive mapping. Let $q, p \in I$, such that $q\Re p$ and $q(\mathfrak{t}) \le p(\mathfrak{t})$ for all $\mathfrak{t} \in [0,1]$. Note that

$$|\mathcal{H}q(\mathfrak{t}) - \mathcal{H}p(\mathfrak{t})| \le \frac{1}{\Gamma(\beta)}\int_0^t (\mathfrak{t}-s)^{\beta-1}\big|f\big(s,q(s)\big) - f\big(s,p(s)\big)\big|ds + \frac{\mathfrak{t}^2}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}\big|f\big(s,q(s)\big) - f\big(s,p(s)\big)\big|ds$$

$$\le \frac{1}{\Gamma(\beta)}\int_0^t (\mathfrak{t}-s)^{\beta-1}\big|f\big(s,q(s)\big) - f\big(s,p(s)\big)\big|ds + \frac{1}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}\big|f\big(s,q(s)\big) - f\big(s,p(s)\big)\big|ds$$

$$\le \frac{1}{\Gamma(\beta)}\int_0^t (\mathfrak{t}-s)^{\beta-1}\lambda|q(s) - p(s)|ds + \frac{1}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}\lambda|q(s) - p(s)|ds$$

$$\le \frac{1}{\Gamma(\beta)}\int_0^t (\mathfrak{t}-s)^{\beta-1}\lambda\|q-p\|_\infty ds + \frac{1}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}\lambda\|q-p\|_\infty ds$$

$$\le \frac{\lambda\|q-p\|_\infty}{\Gamma(\beta)}\int_0^t (\mathfrak{t}-s)^{\beta-1}ds + \frac{\lambda\|q-p\|_\infty}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}ds$$

$$\le \lambda\left(\frac{1}{\Gamma(\beta+1)} + \frac{1}{2\Gamma(\beta-1)}\right)\|q-p\|_\infty$$

$$= \frac{1}{\mathfrak{k}}\|q-p\|_\infty$$

Thus,

$$\mathfrak{k}\|\mathcal{H}q - \mathcal{H}p\|_\infty \le \|q-p\|_\infty$$

This implies

$$e^{-\frac{\mathfrak{k}\|\mathcal{H}q - \mathcal{H}p\|_\infty}{t}} \ge e^{-\frac{\mathfrak{k}\|q-p\|_\infty}{t}}$$

As a result,

$$\varphi\big(\mathcal{M}(\mathcal{H}q, \mathcal{H}p, t)\big) \ge \mathcal{M}(q,p,t) \ge min\{\mathcal{M}(q,p,t), \mathcal{M}(\mathcal{H}q,q,t), \mathcal{M}(p,\mathcal{H}p,t)\},$$

$$\varphi\big(\mathcal{N}(\mathcal{H}q,\mathcal{H}p,t)\big) \leq \mathcal{N}(q,p,t) \leq max\{\mathcal{N}(q,p,t),\mathcal{N}(\mathcal{H}q,q,t),\mathcal{N}(p,\mathcal{H}p,t)\},$$

With $\varphi(t) = t^{k}$ and $k > 1$, this demonstrates that $\mathcal{H}$ is a GV-intuitionistic fuzzy $\Re - \varphi$ −contractive mapping. From (ii), we deduce that $q_0(t)\Re\mathcal{H}q_0(t)$ for all $t \in [0,1]$, hence $q_0\Re\mathcal{H}q_0$, which satisfies condition (i) of Theorem 3. Given $q,p \in I$ with $q(t) \leq p(t)$ for all $t \in [0,1]$, and from (iii), considering f is nondecreasing in the second variable, we obtain

$$\begin{aligned}\mathcal{H}q(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f\big(s,q(s)\big)ds + c_0 + c_0^* t + c_2 t^2\\ &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f\big(s,p(s)\big)ds + c_0 + c_0^* t + c_2 t^2\\ &= \mathcal{H}p(t)\end{aligned}$$

We deduce that $\mathcal{H}q(t) \leq \mathcal{H}p(t)$ for all $t \in [0,1]$, and consequently $\mathcal{H}q \leq \mathcal{H}p$ (i.e., $q\Re p \Rightarrow \mathcal{H}q\Re\mathcal{H}p$). Therefore, $\Re$ is $\mathcal{H}$-closed, satisfying condition (iii) of Theorem 3. Hence, all the hypotheses of Theorem 3 are true, implying that $\mathcal{H}$ has a fixed point which is a solution to Equation (27) in I. Furthermore, if $q,p \in I$ are two fixed points of $\mathcal{H}$ in I, then $q \leq \max\{q,p\}, ,p \leq \max\{q,p\}$, and $r = \max\{q,p\} \in I$. Additionally, $\mathcal{M}(q,r,t) > 0, \mathcal{N}(q,r,t) < 1$ and $\mathcal{M}(q,p,t) > 0$, $\mathcal{N}(q,p,t) < 1$ for all $t > 0$ (according to Definition 4). Therefore, Theorem 4 is also satisfied. Thus, the fixed point of $\mathcal{M}$and $\mathcal{N}$ are unique, making the solution to Equation (27) in I unique. This concludes the proof.

Lastly, we present the following example to support Theorem 5.

Example 3: Let the boundary value problem associated with the fractional differential equation

$$\begin{gathered}\mathcal{D}_{0^+}^{\frac{7}{3}} q(t) = \frac{2q(t)}{3\big(2+q(t)\big)},\quad t \in [0,1]\\ q(0) = 0, q'(0) = 0, q''(1) = 2\end{gathered} \tag{29}$$

Consider the function $f\big(t,q(t)\big) = \frac{2q(t)}{3(2+q(t))}$, $\big(t,q(t)\big) \in [0,1] \times [0,\infty)$

Consider $q(t), p(t) \in [0,\infty)$ and $t \in [0,1]$. Then

$$\begin{aligned}\big|f\big(t,q(t)\big) - f\big(t,p(t)\big)\big| &= \frac{2}{3}\left|\frac{q(t)}{2+q(t)} - \frac{p(t)}{2+p(t)}\right|\\ &= \frac{2}{3}\left|\frac{q(t)-p(t)}{(2+q(t))(2+p(t))}\right|\\ &\leq \frac{2}{3}|q(t)-p(t)|\end{aligned}$$

Therefore, condition (i) of Theorem 5 is fulfilled with $\lambda = \frac{2}{3}$.

Next, we verify that $\lambda\left[\frac{1}{\Gamma(\beta+1)} + \frac{1}{2\Gamma(\beta-1)}\right] < 1$.

$$\frac{2}{3}\left[\frac{1}{\Gamma\left(\frac{10}{3}\right)} + \frac{1}{2\Gamma\left(\frac{4}{3}\right)}\right] = \frac{17}{9\sqrt{\pi}} < 1$$

Therefore, equation (28) is valid. By setting $x_0 = 0$, we then have

$$0 \leq \frac{1}{\Gamma\left(\frac{7}{3}\right)}\int_0^t (t-s)^{\frac{4}{3}} f(s,0)ds - \frac{t^2}{2\Gamma\left(\frac{1}{3}\right)}\int_0^t (1-s)^{-\frac{2}{3}} f(s,0)ds + \frac{t^2}{2} = \frac{t^2}{2},\quad t \in [0,1]$$

This demonstrates that condition (ii) of Theorem 5 is satisfied. Moreover, if $q < p$, we deduce that $f(q) \leq f(p)$. Consequently, condition (iii) of Theorem 5 is also satisfied. Therefore, Equation (29) has a unique solution over the interval [0,1].

5. Conclusion:

This research has introduced and explored the concept of intuitionistic fuzzy R-ψ-contractive mappings, yielding significant results on the existence and uniqueness of fixed points in the context of non-Archimedean intuitionistic fuzzy metric spaces. Our findings extend and generalize existing results, providing a more comprehensive framework for understanding fixed point theory in intuitionistic fuzzy metric spaces. The illustrative examples presented in this paper demonstrate the applicability of our results. Furthermore, we have successfully applied our theoretical findings to establish the existence and uniqueness of solutions for Caputo fractional differential equations in the setting of intuitionistic fuzzy metric spaces.

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