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Research Article

Solution of Various Kinds of Volterra Integral Equations Using Different Methods

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ABSTRACT

Integral equations are widely recognized as powerful tools in the field of applied mathematics. The numerous types of Volterra Integral equation and their solutions using various approaches are covered in this study.The series form demonstrates the solutions' existence and uniqueness. Analytical, convergent, and approximate solutions to the Volterra integral equations are obtained by solving a few cases.

Keywords: Differential equation (DE),Ordinary differential equation (ODE),integral equation (IE), Volterra integral equations(VIE), Successive approximation method (SAM), boundary value problem (BVP) and Variational iteration method (VIM).

INTRODUCTIONS:

Many problems in mathematical physics [1], theory of elasticity [2,3,4], hydrodynamics [5], quantum mechanics [6,7], and contact problems in the theory of elasticity [8,9], take the form of IVPs or BVPs. Numerous branches of various sciences are closely related to the theory of IE. Because of these new problems researchers have developed alternative techniques for resolving different types of IEs. It is possible to convert many initial and boundary value problems related to ODE and PDE into problems involving the solution of certain approximate IEs. The theory of the Fourier Integral introduced us to IEs. Another IE was developed by Abel in 1826. It was the Italian mathematician V. Volterra and the Swedish mathematician Fredholm who initially developed the theory of IEs.

Definition: IE

An IE is an equation that appears under one or more integral symbols and has a function that is not known

$$u(\propto) = f(\propto) + \lambda \int_a^b K(\propto, \tau) y(\tau) d\tau$$

where $y(\tau)$ is the not known function, a and b represent constants, while $f(\propto)$ are known functions. A known function is $K(\propto, \tau)$.

VIE:

VIE of the first kind:

An IE of linear form expressed as follows.

$$\mathrm{f}(\propto) + \lambda \int_{\mathrm{a}}^{\mathrm{x}} \mathrm{M}(\propto, \tau)\, \mathrm{y}(\tau) \mathrm{d}\tau = 0$$

is called VIE of the first kind

VIE of the second kind:

$$y(\propto) = f(\propto) + \lambda \int_a^x N(\propto, \tau)\, y(\tau) d\tau$$

is called VIE of the second kind

VIE of the Third kind:

An IE of linear form expressed as follows.

$$g(\propto)y(\propto) = f(\propto) + \lambda \int_a^x N(\propto, \tau) y(\tau) d\tau$$

Where $g(\propto), f(\propto) and\ k(\propto, t)$ are defined functions, while $y(\propto)$ is an unidentified function. A non-zero real or complex parameter is referred to as the Volterra IE of the third kind. The function $N(\propto, \tau)$ is recognized as the kernel of the IE

VIE of the Fourth kind:

A linear IE of the form

$$u(\propto) = y(\propto) + \int_a^x (\propto - \tau)^n u(\tau)\, d\tau, \qquad t > 0$$

SAM: VIE of the second kind

$$m(\propto) = f(\propto) + \lambda \int_0^x N(\propto, \tau). m(\tau) d\tau \qquad (1)$$

$f(\propto)$ is continuous in$[0, a]$ and $K(\propto, \tau)$ is continuous for $0 \leq \propto \leq a$, $0 \leq t \leq \propto$.

Given function $m_0(x)$ continuous in$[0, a]$.then, replace by $m(\tau)$ on R.H.S. of equation (1) by $m_0(\tau)$, we have

$$m_1(\propto) = f(\propto) + \lambda \int_0^x N(\propto, \tau). m_0(\tau) d\tau \qquad (2)$$

But $m_1(\propto)$ is not discontinuous in $[0, a]$.

$$m_2(\propto) = f(\propto) + \lambda \int_0^x N(\propto, \tau). m_1(\tau) d\tau \qquad (3)$$

But $m_2(\propto)$ is not discontinuous in $[0, a]$.

We take a similar approach and arrive at a series of functions.

$$m_3(\propto), m_4(\propto), \dots, m_n(\propto), \dots$$

$$m_n(\propto) = f(\propto) + \lambda \int_0^x N(\propto, \tau). m_{n-1}(\tau) d\tau \qquad (4)$$

$K(\propto, t)$, the sequences $\{m_n(\propto)\}$ converges as $n \to \infty$

That is $m(\propto) = \lim_{n \to \infty} m_n(\propto)$

VIM for system of ODE:

Consider the general DE

$$P[u(\propto, t)] + M[u(\propto, t)] = g(\propto, t) \qquad (5)$$

Where M is a nonlinear operator, P is linear operator, $g(\propto, t)$ and is in homogeneous function. According to VIM we construct a correction functional $u(t)$,

$$V_{n-1}(\propto, t) = V_n(\propto, t) + \int_0^t \lambda\ (L\, V_n(\propto, s) + M\, \widetilde{v_n}(\propto, s) - g(\propto, s)) ds \qquad (6)$$

In this context,$\widetilde{v_n}(\propto, t)$ represents a restricted variation. The optimal value of the General Lagrange multiplier is identified by applying the stationary condition from variational theory.

Differential Transform Method:

The foundation of the DTM was conceived in 1986 by Zhou (Zhou, 1986). He tackled initial value problems, both linear and nonlinear, that arise in circuit analysis by applying this transform. The method resembles Taylor's series and produces solutions in the form of polynomials. The basic concept of the DTM is outlined below [6].

Definition:k^{th} Order (DT) of a function $y(\propto) = f(\propto)$ is defined at a point $\propto=\propto_0$ as,

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(\propto)}{d\propto^k}\right]_{\propto=\propto_0} \quad (7)$$

$$y(\propto) = \sum_{k=0}^{k=\infty}(\propto - \propto_0)^k Y(k)\,. \quad (8)$$

$$y(\propto) = \sum_{k=0}^{k=\infty}\frac{(\propto-\propto_0)^k}{k!}\left\{\frac{d^k y(\propto)}{dx^k}\right\} \quad (9)$$

An Equation (9) implies that the concept of DTM is derived from Taylor's series expansion. Let us assume that the differential transform of the functions $q(\xi)$,$r(\xi)$ and $s(\xi)$ are given by $Q(p)$ $R(p)$ and $S(p)$ respectively. Differential transform for the standard functions of one variable (Zhou, 1986) are reported in Table 1

Table 1. DT

Sr. No.	Original Function	Transformed function
1	$\psi(\xi) = q(\xi) \pm r(\xi)$	$Q(p) \pm R(p)$
2	$\psi(\xi) = cq(\xi)$	$F(p) = cQ(p)$
3	$\psi(\xi) = q(\xi) \times r(\xi)$	$\sum_{i=0}^{r} A(i) \times B(l-i)$
4	$\psi(\xi) = \frac{d^n(q(\xi))}{d\xi^n}$	$\frac{(p+n)!}{p!}Q(p+n)$
5	$\psi(\xi) = 1$	$\delta(p)$
6	$\psi(\xi) = \xi^m$	$\delta(p-m)$
7	$\psi(\xi) = e^{\lambda\xi}$	$\frac{\lambda^p}{p!}, \lambda isconstant$
8	$\psi(\xi) = \sin(\omega\xi + \alpha)$	$\frac{\omega^p}{p!}\sin\left(\frac{p\pi}{2} + \alpha\right)$
9	$\psi(\xi) = \cos(\omega\xi + \alpha)$	$\frac{\omega^p}{p!}\cos\left(\frac{p\pi}{2} + \alpha\right)$
10	$\psi(\xi) = x\left(\frac{\xi}{a}\right), a \geq 1$	$\frac{1}{a^p}Q(p)$

Example 1: Consider the VIE $u(\propto) = \propto - \int_0^{\propto}(\propto - \tau)u(\tau)d\tau$ with

$u(\propto) = 0$ Then using successive approximation method, $u_0(\propto) = 0$

Solution: The first approximation is given obtain

when $m = 0$ $u_1(\propto) = \propto - \int_0^{\propto} o.\, d\tau = \propto$

$$m = 1, u_2(\propto) = \propto - \int_0^{\propto} (\propto - \tau) u_1(\tau) d\tau = \propto - \frac{\propto^3}{3!}$$

$$m = 2, u_3(\propto) = \propto - \int_0^{t} (\propto - \tau)(\tau - \frac{\tau^3}{3!})\ d\tau = \propto - \frac{\propto^3}{3!} + \frac{\propto^5}{5!}$$

In general , $u_m(\propto) = \propto - \frac{\propto^3}{3!} + \frac{\propto^5}{5!} \pm \cdots + (-1)^n \frac{\propto^{2n-1}}{(2m-1)!}$

Example 2: VIE of the fourth kind

$$u(\propto) = 1 - \int_0^{\propto} u(\propto) d\tau,\ \ 0 < \tau \le 1\ , u(0) = 2$$

The exact solution is, $u(\propto) = e^{-\tau}$

We convert example in to the ODE form

$$u'(\tau) + u(\tau) = 0, \qquad 0 < \tau \le 1$$

with the general solution using VIM for system of ODE

$$u_{m+1}(\tau) = u_m(\tau) + \int_0^1 \lambda(t,s)\{L y_n(s) + N(y_n(s) - g(s))\} ds$$

$$u(\tau) = u_n(\tau) + \int_0^1 \lambda(t,s)\{u'(s) + u(s)\} ds$$

with the condition $\lambda(t,s) = -1$ this gives

$$u_{m+1}(\tau) = u_m(\tau) - \int_0^1 (x - \tau)\{u'(\tau) + u(\tau)\} ds$$

$$u_0(\propto) = 1$$

Put $m = 0,\ u_1(\propto) = 2 - \propto$

Put $m = 1,\ u(\propto) = 2 - \propto + \frac{\propto^2}{2!}$

Put $m = 2,\ u(\propto) = 2 - \propto + \frac{\propto^2}{2!} - \frac{\propto^3}{3!}$

Example3: VIE of the fourth kind

$$u(\propto) = 1 - \propto + \int_0^{x} u(\tau) d\tau, 0 < \tau \le 1\ , u(0) = u'(0) = 1$$

exact solution is ,$u(\propto) = e^{-\tau}$

We convert example in to the ordinary differential equation form

$$u''(\tau) - u(\tau) = 0\ , \qquad 0 < \tau \le 1$$

with the general solution using VIM for system of ODE,

$$u_{n+1}(\tau) = u_n(\tau) + \int_0^1 \lambda(\tau,s)\{L u_n(s) + N(u_n(s) - g(s))\} ds$$

$$u_{n+1}(\tau) = u_n(\tau) + \int_0^1 \lambda(t,s)\{u''(\tau) - u(\tau)\} ds$$

with the condition $\lambda(\tau,s) = x - 1$ this gives

$$u_{n+1}(\tau) = u_n(\tau) - \int_0^1 (x - \tau)\{u''(\tau) + u(\tau)\} ds$$

$$u_0(x) = 1{-}\propto$$

Put $n = 0, u_1(\propto) = 1{-}\propto$

Put $n = 1,\ u_2(\propto) = 1{-}\propto + \frac{\propto^2}{2!}$

Put $n = 2,\ \ u_3(\propto) = 1{-}\propto + \frac{\propto^2}{2!} - \frac{\propto^3}{3!}$

Example 4: Consider the IE $v(\propto) = 1{+}\propto + \int_0^{\propto}(\propto - \tau)\, v(\tau)d\tau$

Solution: $v(\propto) = 1{+}\propto + \int_0^{\propto}(\propto - \tau)\, v(\tau)d\tau$

Applying Differential transform on both sides

$$D\{v(\propto)\} = D\left\{1{+}\propto + \int_0^{\propto}(\propto - \tau)\, v(\tau)d\tau\right\}$$

$$V(k) = D[1] + D[\propto] + D\left[\int_0^{\propto}(\propto - t)\, v(\tau)d\tau\right]$$

$$V(k) = \delta(k) + \delta(k-1) + \sum_{l=0}^{k-1}\frac{\delta(l-1)V(k-l-1)}{k-l} - \sum_{l=0}^{k-1}\frac{\delta(l-1)V(k-l-1)}{k}, k \geq 1, V(0) = 1$$

$For\ k = 1$

$$V(1) = \delta(1) + \delta(0) + \frac{\delta(-1)U(0)}{1} - \frac{\delta(-1)V(0)}{1}$$

$$V(1) = 1$$

$for\ k = 2\ \ V(2) = \frac{1}{2} = \frac{1}{2!}$

$for\ k = 3,\ \ V(3) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \frac{1}{3!}$

Similarly, $V(4) = \frac{1}{4!}, V(5) = \frac{1}{5!}, V(6) = \frac{1}{6!}$, ...

Therefore,

$$v(\propto) = \sum_{k=0}^{k=\infty}(\propto - \propto_0)^k V(k)$$

$$v(\propto) = 1{+}\propto + \frac{\propto}{2!} + \frac{\propto^3}{3!} + \cdots$$

$$v(\propto) = e^{\propto}$$

CONCLUSION:

In our study we discuss the various methods to find the solutions of the Volterra IEs of various kind. We obtain the solutions in series form .Solutions of the Volterra IEs can be obtain easily, reducing the calculus work by Differential transform method.

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