Journal of Information Systems Engineering and Management
2025, 10 (31s)
e-ISSN: 2468-4376
https://jisem-journal.com/

Research Article

On Class of Area Vanishing Functions on the Unit Interval

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ARTICLE INFO

Received: 22 Dec 2024

Revised: 17 Feb 2025

Accepted: 24 Feb 2025

ABSTRACT

Among several classes of Riemann integrable real valued functions we are interested in finding class of functions whose (signed) integral values over a compact interval vanishes. Determining such class of functions was one of the important objectives of this paper. We begin our quest by introducing Bernoulli numbers then extending them to Bernoulli polynomials. We observe that Bernoulli polynomials are generalized version of the most famous and notorious Bernoulli numbers introduced by Jacob Bernoulli in 1713. In particular, we see that Bernoulli numbers are simply the constant terms of Bernoulli polynomials. Bernoulli numbers and Bernoulli polynomials play very big role in analyzing several aspects of mathematics and they occur unexpectedly in several counting problems. We prove some interesting properties of Bernoulli polynomials which generate another class of functions having the property of zero area in [0,1]. In this paper, we try to establish that such class of functions are precisely the Bernoulli polynomials and prove that the Riemann integral of five categories of Bernoulli polynomials over the compact interval [0,1] is zero. The geometric meaning of this fact for special cases is explained through several figures which will provide better insight and understanding. This paper will also provide an scope for generalizing in the analysis of Riemann integration of Bernoulli polynomials not restricted to just the interval [0,1] but for any compact interval in the real line.

Keywords: Maclaurin's Series Expansions, Bernoulli Numbers, Bernoulli Polynomials, Riemann Integral over a compact interval, Class of Area Vanishing Functions

1. INTRODUCTION

Ever since, Jacob Bernoulli published his phenomenal paper about Bernoulli numbers in 1713, the interest and research about these numbers grew exponentially. Today we have thousands of papers devoted to the study of Bernoulli numbers and its generalizations. In this paper, we shall study about the behavior of integrals of

Bernoulli polynomials over the compact interval [0,1]. First, we shall recall the Bernoulli numbers and Bernoulli polynomials (see **[1]** to **[10]**) to proceed further.

2.1 DEFINITION

Bernoulli numbers are class of numbers obtained as coefficients of $\frac{x^n}{n!}$ in the Maclaruin's series expansion of the function $f(x)=\frac{x}{e^x-1}$.

In particular, the nth Bernoulli number is given by $\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$ (2.1)

By exponential series, we have

$$\frac{x}{e^x-1}=\frac{x}{\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\cdots}$$

$$=1-\frac{1}{2}\frac{x}{1!}+\frac{1}{6}\frac{x^2}{2!}-\frac{1}{30}\frac{x^4}{4!}+\frac{1}{42}\frac{x^6}{6!}-\frac{1}{30}\frac{x^8}{8!}+\frac{5}{66}\frac{x^{10}}{10!}+\cdots \quad (2.2)$$

From equation (2.2) and comparing definition (2.1), we have

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30}, B_5=0, B_6=\frac{1}{42},$$
$$B_7=0, B_8=-\frac{1}{30}, B_9=0, B_{10}=\frac{5}{66}, \cdots \quad (2.3)$$

Equation (2.3) provides us with first ten Bernoulli numbers. We notice that apart from B_1, all odd indexed Bernoulli numbers are zero.

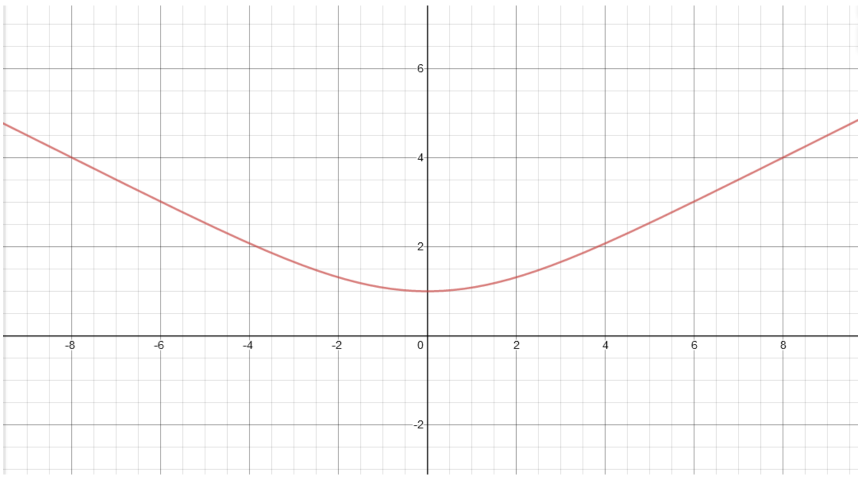

Figure 1: Graph of $f(x)=\frac{x}{e^x-1}-B_1x=\frac{x}{e^x-1}+\frac{x}{2}$

One possible reason for why the odd indexed Bernoulli numbers apart from are all zero is explained in Figure 1, where we see that the function $f(x) = \frac{x}{e^x - 1} - B_1 x = \frac{x}{e^x - 1} + \frac{x}{2}$ is an even function and so there would be no odd powers involved in it.

We now discuss the generalized form of Bernoulli numbers namely Bernoulli polynomials.

3.1 Definition

Bernoulli polynomials are class of polynomials which occur as coefficients of $\frac{x^n}{n!}$ in the Maclaruin's series expansion of the function $g(x) = \frac{xe^{tx}}{e^x - 1}$

In particular, the nth Bernoulli polynomial is given by $\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}$ (3.1)

Upon comparing the equations (2.1) and (3.1), we see that $B_n(0) = B_n$ (3.2)

Thus from equation (3.2), we can understand that the Bernoulli numbers are special class of numbers viewed as constant terms of the Bernoulli polynomials.

3.2 GENERATING BERNOULLI POLYNOMIALS

The nth Bernoulli polynomial can be generated using the identity $B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_k t^{n-1}$ (3.3) where B_k is the kth Bernoulli number. Here $\binom{n}{k}$ is the number of ways of choosing or selecting k among n things. For proof of (3.3), see **[4]** by the corresponding author.

Using equation (3.3), we have the following Bernoulli polynomials.

For $n = 0$, $B_0(t) = B_0 = 1$ (3.4)

For $n = 1$, $B_1(t) = \sum_{k=0}^{1} \binom{1}{k} B_k t^{1-k} = B_0 t + B_1 = t - \frac{1}{2}$ (3.5)

For $n = 2$, $B_2(t) = \sum_{k=0}^{2} \binom{2}{k} B_k t^{2-k} = B_0 t^2 + 2B_1 t + B_2 = t^2 - t + \frac{1}{6}$ (3.6)

For n = 3, $B_3(t)=\sum_{k=0}^{3}\binom{3}{k}B_k t^{3-k} = B_0t^3+3B_1t^2+3B_2t+B_3 = t^3-\frac{3}{2}t^2+\frac{1}{2}t$ (3.7)

Similarly for n = 4, 5 and 6 we get

$$B_4(t)=B_0t^4+4B_1t^3+6B_2t^2+4B_3t+B_4=t^4-2t^3+t^2-\frac{1}{30} \quad (3.8)$$

$$B_5(t)=B_0t^5+5B_1t^4+10B_2t^3+10B_3t^2+5B_4t+B_5=t^5-\frac{5}{2}t^4+\frac{5}{3}t^3-\frac{1}{6}t \quad (3.9)$$

$$B_6(t)=B_0t^6+6B_1t^5+15B_2t^4+20B_3t^3+15B_4t^2+6B_5t+B_6=t^6-3t^5+\frac{5}{2}t^4-\frac{1}{2}t^2+\frac{1}{42} \quad (3.10)$$

Equations (3.4) to (3.10) provide the first seven Bernoulli polynomials. In similar fashion, we can generate Bernoulli polynomials of higher orders. We now prove an important theorem regarding the derivative of Bernoulli polynomials.

4.1 Theorem 1

If $B_n(t)$ is the nth Bernoulli polynomial then $B'_{n+1}(t)=(n+1)B_n(t)$ (4.1)

Proof: First we note that the nth Bernoulli polynomial $B_n(t)$ is a polynomial of degree n. Hence it is continuously differentiable and its nth derivate is $n!$. Hence the differentiation of $B_n(t)$ is well defined. Now differentiating $B_n(t)$ with respect to t, we get

$$\frac{d}{dt}\left(\frac{xe^{tx}}{e^x-1}\right)=\frac{d}{dt}\left(\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}\right)=\sum_{n=1}^{\infty}B'_n(t)\frac{x^n}{n!}$$

$$\frac{x^2e^{tx}}{e^x-1}=\sum_{n=1}^{\infty}B'_n(t)\frac{x^n}{n!}$$

Now dividing both sides by x and replacing n by n + 1 in the summation in the right hand side, we get

$$\frac{xe^{tx}}{e^x-1}=\sum_{n=1}^{\infty}B'_n(t)\frac{x^{n-1}}{n!}=\sum_{n=0}^{\infty}B'_{n+1}(t)\frac{x^n}{(n+1)!}=\sum_{n=0}^{\infty}\frac{B'_{n+1}(t)}{n+1}\frac{x^n}{n!} \quad (4.2)$$

Comparing (3.1) and (4.2), we get $\frac{B'_{n+1}(t)}{n+1}=B_n(t)$. Hence, we get

$$B'_{n+1}(t)=(n+1)B_n(t)$$

which is equation (4.1) as desired.

Since $B_n(t)$ is a polynomial of degree n, $B_n(t)$ is continuous for all t in [0,1]. Hence it is Riemann Integrable in [0,1]. The following theorem provides the integral value of

$B_n(t)$ for all t in [0,1]. In the following theorems we consider n to be a natural number.

4.2 Theorem 2

If $B_n(t)$ is the nth Bernoulli polynomial then $\int_0^1 B_n(t)dt = 0$ (4.3)

Proof: Using equation (4.1) of Theorem 1, we get

$$\int_0^1 B_n(t)dt = \int_0^1 \frac{B'_{n+1}(t)}{n+1}dt = \frac{1}{n+1}\left[B_{n+1}(t)\right]_{t=0}^1 = \frac{1}{n+1}\left[B_{n+1}(1) - B_{n+1}(0)\right]$$

We note that $B_{n+1}(0) = B_{n+1}$ is the (n+1)th Bernoulli number.

From **[4]**, we note that the Bernoulli numbers B_k satisfy $\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0$.

From equation (3.3), we get

$$B_{n+1}(1) = \sum_{k=0}^{n+1}\binom{n+1}{k}B_k = \sum_{k=0}^{n}\binom{n+1}{k}B_k + B_{n+1} = 0 + B_{n+1} = B_{n+1}$$

Using these values, we get $\int_0^1 B_n(t)dt = \frac{1}{n+1}\left[B_{n+1} - B_{n+1}\right] = 0$

This is equation (4.3), completing the proof.

Note that we have made use of Fundamental Theorem of calculus in this proof when we integrated the derivative of $B_n(t)$. Further from Theorem 2, we can conclude that the integral value of Bernoulli polynomials $B_n(t)$ over the compact interval [0,1] is zero. We now try to find other class of polynomials whose integral value also vanishes over [0,1]. For doing this, we need the following theorem.

5.1 Theorem 3

The Bernoulli polynomials $B_n(t)$ satisfy the relation $B_n(1-t) = (-1)^n B_n(t)$ (5.1)

Proof: Using the equation (3.1), we get

$$\sum_{n=0}^{\infty}(-1)^n B_n(t)\frac{x^n}{n!} = \sum_{n=0}^{\infty}B_n(t)\frac{(-x)^n}{n!} = \frac{-xe^{-tx}}{e^{-x}-1}$$

$$= \frac{-xe^{-tx}}{e^{-x}-1} \times \left(\frac{-e^x}{-e^x}\right) = \frac{xe^{-tx}e^x}{e^x-1}$$

$$= \frac{xe^{(1-t)x}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(1-t)\frac{x^n}{n!}$$

Comparing the coefficients of $\frac{x^n}{n!}$ on both sides we get $B_n(1-t) = (-1)^n B_n(t)$ as desired.

With the aid of Theorem 3, we prove the following important theorem.

5.2 Theorem 4

If $B_n(t)$ is the nth Bernoulli polynomial then $\int_0^1 B_n(1-t)dt = 0$ (5.2)

Proof: Using equation (4.3) of Theorem 2 and (5.1) of Theorem 3, we get

$$\int_0^1 B_n(1-t)dt = \int_0^1 (-1)^n B_n(t)dt = (-1)^n \int_0^1 B_n(t)dt = 0$$

This completes the Proof.

We now prove a property of Bernoulli polynomials which will help us in further exploration.

6.1 Theorem 5

The nth Bernoulli polynomial $B_n(t)$ satisfies the relation $B_n(t+1) - B_n(t) = nt^{n-1}$ (6.1)

Proof: Using equation (3.1), we have

$$\sum_{n=0}^{\infty}\left[B_n(t+1) - B_n(t)\right]\frac{x^n}{n!} = \sum_{n=0}^{\infty} B_n(t+1)\frac{x^n}{n!} - \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!} = \frac{xe^{(t+1)x}}{e^x - 1} - \frac{xe^{tx}}{e^x - 1}$$

$$= \frac{xe^{tx}\left(e^x - 1\right)}{e^x - 1} = xe^{tx} = \sum_{n=0}^{\infty} x\frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} t^n \frac{x^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} (n+1)t^n \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} nt^{n-1}\frac{x^n}{n!}$$

Comparing the coefficients of $\frac{x^n}{n!}$ on both sides we get $B_n(t+1) - B_n(t) = nt^{n-1}$ as desired.

Using equation (6.1), we can generate several families of polynomials whose area vanishes over the interval [0,1] as proved in the following two theorems.

6.2 Theorem 6

If $B_n(t)$ is the nth Bernoulli polynomial then $\int_0^1 \left[1 - B_n(t+1)\right]dt = 0$ (6.2)

Proof: Using equation (6.1) of Theorem 5 and equation (4.3) of Theorem 2, we have

$$\int_0^1 \left[1 - B_n(t+1)\right] dt = \int_0^1 \left[1 - B_n(t) - nt^{n-1}\right] dt = 1 - \int_0^1 B_n(t)\,dt - \left[t^n\right]_{t=0}^{1}$$
$$= 1 - 0 - 1 = 0$$

This completes the proof.

6.3 Theorem 7

If $B_n(t)$ is the nth Bernoulli polynomial then

$$\int_0^1 \left[B_{2n-1}(t+1) + B_{2n-1}(t-1)\right] dt = 0 \qquad (6.3)$$

$$\int_0^1 \left[B_{2n}(t+1) - B_{2n}(t-1)\right] dt = 0 \qquad (6.4)$$

Proof: From (6.1), we have $B_k(t+1) - B_k(t) = kt^{k-1}$.

Replacing t + 1 by t we have

$$B_k(t) - B_k(t-1) = k(t-1)^{k-1}$$

From these two equations, we get

$$\int_0^1 \left[B_k(t+1) + B_k(t-1)\right] dt = \int_0^1 \left[kt^{k-1} + B_k(t) + B_k(t) - k(t-1)^{k-1}\right] dt$$
$$= k\int_0^1 \left[t^{k-1} - (t-1)^{k-1}\right] dt + 2\int_0^1 B_k(t)\,dt$$
$$= \left[t^k - (t-1)^k\right]_{t=0}^{1} + 2(0) = 1 + (-1)^k$$

Now if k is odd then $1 + (-1)^k = 0$. Thus we obtain equation (6.3) by considering $k = 2n - 1$.

In similar fashion, we obtain

$$\int_0^1 \left[B_k(t+1) - B_k(t-1)\right] dt = \int_0^1 \left[kt^{k-1} + B_k(t) - B_k(t) + k(t-1)^{k-1}\right] dt$$
$$= k\int_0^1 \left[t^{k-1} + (t-1)^{k-1}\right] dt = \left[t^k + (t-1)^k\right]_{t=0}^{1}$$
$$= 1 - (-1)^k$$

If k is even, then $1 - (-1)^k = 0$, giving equation (6.4) by considering $k = 2n$.

This completes the proof.

7.1 We now consider certain figures to verify the results obtained in the theorems proved above.

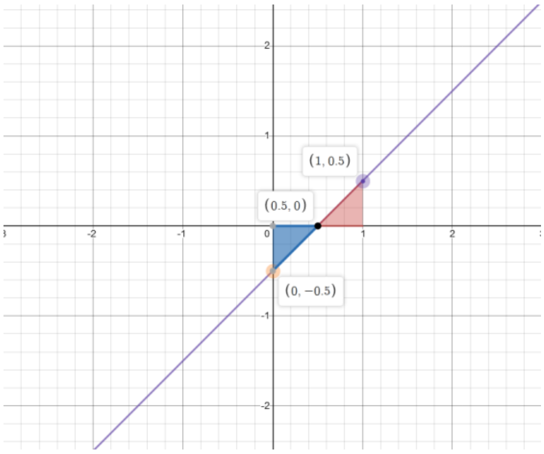

Figure 2: Area bounded by $B_1(t)$ in [0,1]

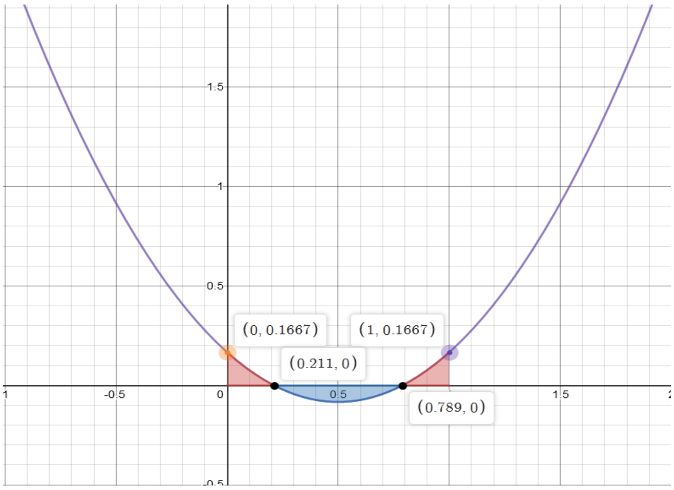

Figure 3: Area bounded by $B_2(t)$ in [0,1]

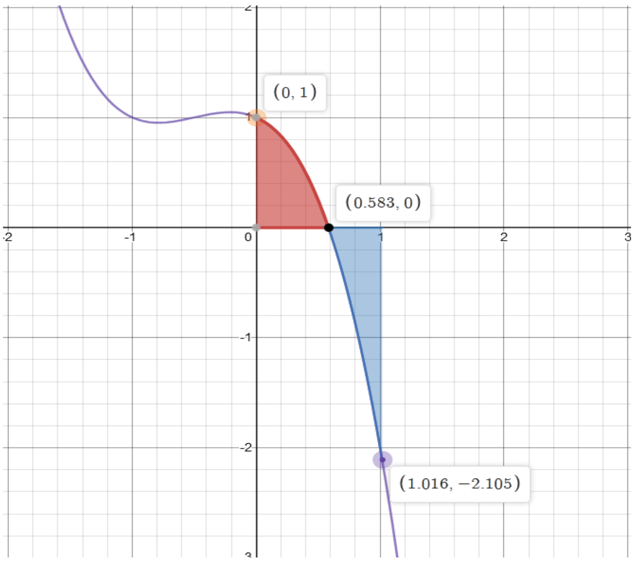

Figure 4: Area bounded by $B_3(1- t)$ in [0,1]

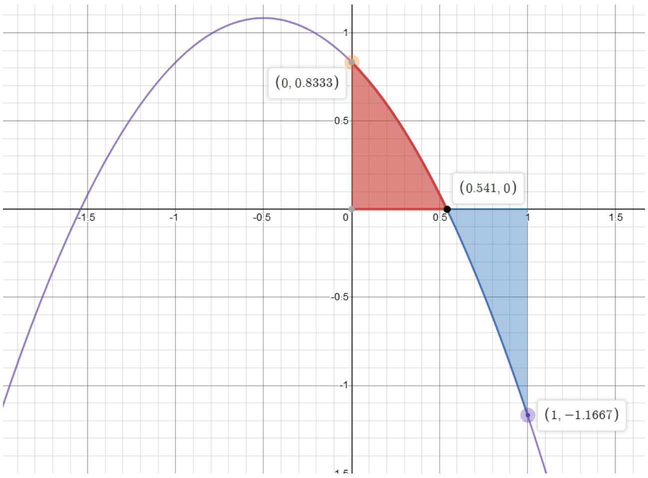

Figure 5: Area bounded by 1- $B_2(t+1)$ in [0,1]

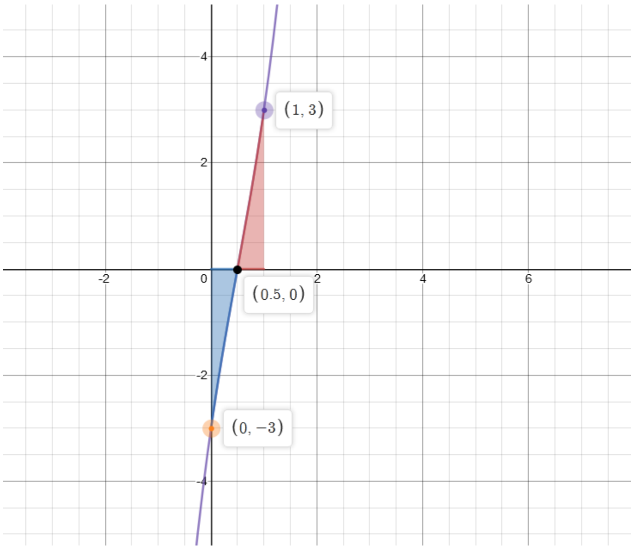

Figure 6: Area bounded by $B_3(t+1)+ B_3(t-1)$ in [0,1]

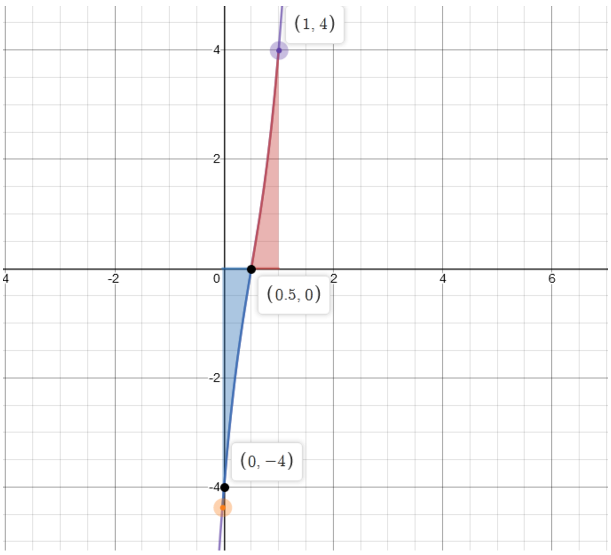

Figure 7: Area bounded by $B_4(t+1)$ - $B_4(t-1)$ in [0,1]

The figures from 2 to 7 confirm the fact that the area bounded by respective functions in the interval [0,1] is zero because the area bounded in blue region is equal and opposite to that of in red region. These observations verifies the results obtained in theorems 2, 4, 6 and 7.

8.1 CONCLUSION

Among several class of functions that exist whose signed area vanish over compact intervals over the real line, this paper focuses on five classes of functions through Bernoulli polynomials which all vanish over the interval [0,1]. Since Bernoulli polynomials of all orders are continuous they are Riemann integrable and so the process of getting zero signed area is well defined.

In particular, in this paper, we have shown that the five class of functions $B_n(t), B_n(1-t), 1-B_n(t+1), B_{2n-1}(t+1)+B_{2n-1}(t-1), B_{2n}(t+1)-B_{2n}(t-1)$ all have zero signed area over [0,1] through theorems 2, 4, 6 and 7 respectively. To have better understanding of these results, we have provided six Figures from Figure 2 to Figure 7 verifying them for few special cases of functions discussed in the theorems. This will provide an visual insight of the truths established. In particular, by viewing the fact each contain an region in blue which has equal but opposite area to that of the region marked in red will confirm that the signed area of all the functions considered in the interval [0,1] must be zero. This paper, thus, provides us with the idea of class of functions via Bernoulli polynomials whose areas vanish under the compact interval [0,1].

We can extend this idea to analyze the area values for other compact intervals in the real line or some special intervals of the form say $[-a, a]$ or $[0, a]$ for some positive real number a. This will provide a new insight for further investigation. We can also consider several other class of functions and apply Integral Transforms like Laplace Transforms, Fourier Transforms etc and see what kind of functions have zero area in a particular compact interval in the real line of the form $[a, b]$.

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