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Research Article

Stochastic Analysis of Vital Organs Failure and Treatment with AkashSanker Distribution

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ARTICLE INFO | **ABSTRACT**

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This paper assumes that a diabetic person has two defective vital organs kidney and heart. In this research paper we study 4 models in which the patient gets Kidney and Heart ailments. In these models, he is sent for hospital treatment at time T for a Kidney ailment or for a Heart ailment or for preventive purpose (Prophylactic treatment) whichever occurs first. His treatment time is denoted by R which may be either for Kidney or Heart or for preventive purpose as per the requirement. The joint probability distribution of T and R , its pdf and the Laplace transform of the pdf are derived. The expected time E(T) of T , the time to send for hospital treatment and the expected time E(R) of R , the duration of treatment time in the hospital are obtained. Variances of T and R are also derived.

Keywords: Mixture of Exponential distribution, Erlang distribution with phase 3, Erlang distribution with phase 2, Akash Sankar distribution, Gamma distribution.

INTRODUCTION:

India has almost 66.8 million instances of diabetes, and these numbers are supposed to ascend to 120.9 million by 2035 . Type 2 diabetes in Asian Indians is unique and particular in light of multiple factors, calculations created and approved in created countries may not be significant or pertinent to patients in India. It has for some time been perceived that Type 2 diabetes in Asian Indians contrasts essentially from that tracked down in white Caucasians. In agricultural nations, not exactly 50% of individuals with diabetes are analysed. Without convenient judgments and sufficient treatment, entanglements and horribleness from diabetes rise dramatically[17,24,28,29]. The majority of the intricacies of diabetes can be forestalled by early findings and thorough treatment. Ralph A DeFronzo composed a worldwide course book on Diabetes mellitus. Accomplishing glycemic control is the foundation of any diabetes program [1,5,9,13]. Tight glucose control fundamentally diminishes the gamble of creating both miniature and macro vascular confusions[1]. Diabetes mellitus has for some time been perceived as a free gamble factor for a few types of cardiovascularillness in all kinds of people.[1] For sure cardiovascular difficulties are presently the main sources of sickness and demise in diabetic patients and subsequently, there is a rising meaning of diabetes mellitus as a significant gamble factor for cardiovascular illness. Sadikot SM, Nigam A, Das S, Prasannakumar KM, et al. examined the weight of diabetes and disabled glucose resistance in India utilizing the WHO 1999 measures . Vascular gamble increments with all degrees of glucose bigotry, even in those individuals without diabetes. Numerous components for blood vessel harm, for example, endothelial irregularities, vascular penetrability, lipoprotein profile, oxidate pressure, platelet collection factors, coagulation variables, and insulin harshness are impacted by higher blood glucose levels[3,4,10,15,23,24]. With expanding levels of blood glucose, there is a proportionate expansion in degrees of all out cholesterol.[5] Insulin cold-heartedness/insulin obstruction is a diminishing in the viability of insulin or the responsiveness of the body's different biologic cycles to insulin. Heights in flowing insulin levels have been related with an expanded gamble of cardiovascular sickness[8].

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Chowdhury TA and Lasker SS have focused on inconveniences and cardiovascular gamble factors.Akashdistribution , proposed by Shanker, allows us to calculate the mean and variance values [2,21,22]In this paper we study 4 models in which the patient gets Kidney and Heart ailments. In these models, he is sent for hospital treatment at time T for Kidney ailment or for Heart ailment or for preventive purpose (Prophylactic treatment) whichever occurs first. His treatment time is denoted by R which may be either for Kidney or for Heart or for preventive purpose as per the requirement. The joint probability distribution of $T$and R, its pdf and the Laplace transform of the pdf are derived[20,25,26,27]. The expected time $E(T)$ of T the time to send for hospital treatment and the expected time $E(R)$ of R the treatment time required in the hospital are obtained. Variances of T and R are also derived [4,9,10,25,26,27]. The four models treated are listed below.
In Model 1 the Kidney ailment occurs at a random time with distribution which is a mixture of Exponential distribution with parameter λ and Erlang distribution with phase 3 and scale parameter ϑ where the probabilities of the mixture are α and $\beta = 1\alpha$[11,16]. The Heart ailment occurs at random time which has Erlang distribution with phase 2 and scale parameter a. The time to prophylactic treatment has general distribution. All treatment timesassumed are general.
In Model 2, we treat Model 1 with Akashshanker mixture for time to Kidney ailment for treatment where the scale parameter is only ϑ for both Exponential and Erlang phase 3 distributions and the probabilities for mixture are $\alpha = \left(\frac{\vartheta^2}{\vartheta^2+2}\right)$ and $\beta = \left(\frac{2}{\vartheta^2+2}\right)$, where $\alpha + \beta = 1$. In Model 3, we treat Model 1 with exponential Prophylactic time distribution with parameter b.
In Model 4, we treat Model 2 with exponential Prophylactic time distribution with parameter b.

Section 1:Model 1 With Mixture of $\mathrm{E}(\lambda)$ and $\mathrm{E}((3,\vartheta)$ forKidney Ailment Time for Treatment

Assumptions:
(I) Kidney gets affected at a random time following probability distribution which is a mixture of exponential distribution with parameter λ with probability α and Erlang distribution with phase 3 and parameter ϑ with probability $\beta = 1 - \alpha$. This mixture has Cdf is

$$F_k(x) = \alpha\left(1 - e^{-\lambda x}\right) + \beta\left(1 - e^{-\vartheta x} - \vartheta x e^{-\vartheta x} - \frac{1}{2}\vartheta^2 x^2 e^{-\vartheta x}\right)$$

with pdf

$$f_k(x) = \alpha\left(\lambda e^{-\lambda x}\right) + \beta\left(\vartheta^3 \frac{x^2}{2} e^{-\vartheta x}\right). \tag{1}$$

(II) Heart gets affected in a random time following Erlang distribution with phase 2 and with scale parameter a whose pdf is $f_h(x) = \frac{a^2 x e^{-ax}}{1} = a^2 x e^{-ax}$ with Cdf

$$F_h(x) = 1 - e^{-ax} - axe^{-ax} \text{ where } x, a > 0. \tag{2}$$

(III) The patient is also provided prophylactic treatment at a random time following general distribution with pdf $f_p(x)$ and with Cdf$F_p(x)$.

(IV) At time 0 there is no damage to any organ and the random times described above are independent of each other.

(V) The patient is referred for treatment upon whichever event happens first: organ failure or prophylactic purpose.

(VI) The treatment time for Kidney, Heart or for Prophylactic purpose are independent with pdf and Cdf which are respectively $(r_k(y), R_k(y)), (r_h(y), R_h(y))$ and $\left(r_p(y), R_p(y)\right)$.

(VII) In this article we use Akash Sankar (AS) distribution for organ Kidney failure time. It may be noted that Exponential distribution has constant failure rate where as it is better to consider distributions with failure rate depending on time. Erlang order 2 and Akashshanker have failure rates which are not constant.

Analysis:

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Considering the three failures and three treatments we note the joint pdf of T the time to send for treatment to hospital and R the hospital treatment time for the patient is

$$f_{T,R}(x,y) = f_k(x)\bar{F}_h(x)\bar{F}_p(x)r_k(y) + \bar{F}_k(x)f_h(x)\bar{F}_p(x)r_h(y) + \bar{F}_k(x)\bar{F}_h(x)f_p(x)r_p(y) \tag{3}$$

where $\bar{G}_L(x) = 1 - G_L(x)$ where $G_L(x)$ is any distribution function. The initial term on the right-hand side of (3) is the pdf part that Kidney ailment time is of length x, ailment time of Heart and Prophylactic time are of lengths more than x and the Kidney treatment time is of length y. The second term of the right side of (3) is the pdf part that Heart ailment time is of length x, ailment time of Kidney and Prophylactic time are of lengths more than x and the Heart treatment time is of length y. The third term of the right side of (3) is the pdf part that Prophylactic time is of length x, the times to send the patient for the treatment of Kidney or Heart are of lengths more than x and theProphylactic treatment time is of length y. Using equations (1) to (3) we note the joint pdf is

$$\begin{aligned} f_{T,R}(x,y) =& \left(\alpha\lambda e^{-\lambda x} + \beta\vartheta^3\frac{x^2}{2}e^{-\vartheta x}\right)(e^{-ax} + axe^{-ax})\bar{F}_p(x)r_k(y) \\ &+\left\{\alpha e^{-\lambda x} + \beta e^{-\vartheta x}\left(1+\vartheta x+\frac{1}{2}\vartheta^2x^2\right)\right\}(e^{-ax} + axe^{-ax})f_p(x)r_p(y) \\ &+\left\{\alpha e^{-\lambda x} + \beta e^{-\vartheta x}\left(1+\vartheta x+\frac{1}{2}\vartheta^2x^2\right)\right\}(e^{-ax} + axe^{-ax})f_p(x)r_p(y) \end{aligned} \tag{4}$$

The left side pdf is split into three functions. The first part of the pdf is that of Kidney gets ailment first at x with its treatment completes at y which is the first term of right-hand side. The second part of the pdf is that of Heart gets ailment first at x with its treatment completes at ywhich is the second term of right-hand side. The third part of the pdf is that of Prophylactic treatment starts first at x with its treatment completes at y which is the third term of right-hand side. The Laplace transform of the joint pdf of (T, R) is

$$\begin{aligned} f^*_{T,R}(\xi,\eta) =& \iint_0^\infty e^{-\xi x}e^{-\eta y}f_{T,R}(x,y)dxdy \\ =& \int_0^\infty\int_0^\infty e^{-\xi x}\,e^{-\eta y}\left[\left(\alpha\lambda e^{-\lambda x} + \beta\vartheta^3\frac{x^2}{2}e^{-\vartheta x}\right)(e^{-ax} + axe^{-ax})\bar{F}_p(x)r_k(y)\right. \\ &+\left\{\alpha e^{-\lambda x} + \beta e^{-\vartheta x}\left(1+\vartheta x+\frac{1}{2}\vartheta^2x^2\right)\right\}a^2xe^{-ax}\bar{F}_p(x)r_h(y) \\ &\left.+\left\{\alpha e^{-\lambda x} + \beta e^{-\vartheta x}\left(1+\vartheta x+\frac{1}{2}\vartheta^2x^2\right)\right\}(e^{-ax} + axe^{-ax})f_p(x)r_p(y)\right]dxdy. \end{aligned} \tag{5}$$

where * indicates Laplace transform. This gives

$$\begin{aligned} f^*_{T,R}(\xi,\eta) =& \int_0^\infty e^{-\xi x}\left(\alpha\lambda e^{-\lambda x} + \beta\vartheta^3\frac{x^2}{2}e^{-\vartheta x}\right)(e^{-ax} + axe^{-ax})\bar{F}_p(x)dx r^*_k(n) \\ &+\int_0^\infty e^{-\xi x}\left\{\alpha e^{-\lambda x} + \beta e^{-\vartheta x}\left(1+\vartheta x+\frac{1}{2}\vartheta^2x^2\right)\right\}a^2xe^{-ax}\bar{F}_p(x)dx r^*_h(\eta) \\ &+\int_0^\infty e^{-\xi x}\left(\alpha\lambda e^{-\lambda x} + \beta\vartheta^3\frac{x^2}{2}e^{-\vartheta x}\right)(e^{-ax} + axe^{-ax})\bar{F}_p(x)dx\, r^*_p(\eta) \end{aligned} \tag{6}$$

We evaluate the integrals one by one.
(i) Now we see the integral

$$\int_0^\infty e^{-\xi x}\left(\alpha\lambda e^{-\lambda x} + \beta\vartheta^3\frac{x^2}{2}e^{-\vartheta x}\right)(e^{-ax} + axe^{-ax})\bar{F}_p(x)dx$$

$$\begin{aligned} &= \int_0^\infty e^{-x(\xi+\lambda+a)}\alpha\lambda(1+ax)\bar{F}_p(x)dx + \int_0^\infty e^{-x(\xi+\vartheta+a)}\frac{\beta\vartheta^3}{2}(x^2+ax^3)\bar{F}_p(x)dx \\ &= \alpha\lambda\left[\bar{F}_p^{\,*}(\xi+\lambda+a) - a\,\bar{F}_p^{\,*}(\xi+\lambda+a)\right] + \frac{\beta\vartheta^3}{2}\left[\bar{F}_p^{\,*''}(\xi+\lambda+a) - a\,\bar{F}_p^{\,*'''}(\xi+\lambda+a)\right] \end{aligned} \tag{7}$$

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Here I indicates differentiation with respect to ξ.
(ii) Now we see the integral

$$\int_0^\infty e^{-\xi x}\left\{\alpha e^{-\lambda x}+\beta e^{-\vartheta x}\left(1+\vartheta x+\frac{1}{2}\vartheta^2x^2\right)\right\}a^2xe^{-ax}\bar{F}_p(x)dx$$
$$=\int_0^\infty e^{-\xi x}\alpha e^{-\lambda x}a^2xe^{-ax}\bar{F}_p(x)dx$$
$$+\int_0^\infty e^{-\xi x}\beta e^{-\vartheta x}\left(1+\vartheta x+\frac{1}{2}\vartheta^2x^2\right)\Big\}a^2xe^{-ax}\bar{F}_p(x)dx$$
$$=-\alpha a^2\bar{F}_p^{*'}(\xi+\lambda+a)$$
$$+\beta a^2\left[-\bar{F}_p^{*'}(\xi+\vartheta+a)+\vartheta\,\bar{F}_p^{*'}(\xi+\vartheta+a)-\frac{1}{2}\vartheta^2\bar{F}_p^{*'''}(\xi+\vartheta+a)\right] \tag{8}$$

(iii) Now we see the integral

$$\int_0^\infty e^{-\{x}\left\{ae^{-\lambda x}+\beta e^{-\vartheta x}\left(1+\vartheta x+\frac{1}{2}\vartheta^2x^2\right)\right\}(e^{-ax}+axe^{-ax})f_p(x)dx$$
$$=\int_0^\infty e^{-\xi x}\alpha e^{-\lambda x}(e^{-ax}+axe^{-ax})f_p(x)dx$$
$$+\int_0^\infty e^{-\xi x}\beta e^{-\vartheta x}\left(1+\vartheta x+\frac{1}{2}\vartheta^2x^2\right)\Big\}(e^{-ax}+axe^{-ax})f_p(x)dx \tag{9}$$
$$=\alpha\left[f_p^*(\xi+\lambda+a)-af_p^{*'}(\xi+\lambda+a)\right]$$
$$+\beta\left[f_p^*(\xi+\vartheta+a)-(\vartheta+a)f_p^{*'}(\xi+\vartheta+a)+\left(\frac{\vartheta(\vartheta+2a)}{2}\right)f_p^{*''}(\xi+\vartheta\right.$$
$$\left.+a)-\frac{1}{2}a\vartheta^2f_p^{*'''}(\xi+\vartheta+a)\right].$$

Also noting the relation between survivor function and density function we find the transforms satisfies for both ϑ and λ that

$$\bar{F}_p^*(\xi+\vartheta+a)=\frac{1-f_p^*(\xi+\vartheta+a)}{\xi+\vartheta+a}. \tag{10}$$

So

$$1-f_p^*(\xi+\vartheta+a)=(\xi+\vartheta+a)\bar{F}_p^*(\xi+\vartheta+a). \tag{11}$$

Using differentiation of both sides of (12) we note that

$$-f_p^{*\prime}(\xi+\vartheta+a)=\bar{F}_p^*(\xi+\vartheta+a)+(\xi+\vartheta+a)\bar{F}_p^{*\prime}(\xi+\vartheta+a). \tag{12}$$

Differentiating again

$$-f_p^{*\prime\prime}(\xi+\vartheta+a)=2\bar{F}_p^{*\prime}(\xi+\vartheta+a)+(\xi+\vartheta+a)\bar{F}_p^{*\prime\prime}(\xi+\vartheta+a). \tag{13}$$

Differentiating again

$$-f_p^{*\prime\prime\prime}(\xi+\vartheta+a)=3\bar{F}_p^{*\prime\prime}(\xi+\vartheta+a)+(\xi+\vartheta+a)\bar{F}_p^{*\prime\prime\prime}(\xi+\vartheta+a). \tag{14}$$

This gives using(7),(8),(9) and (10)

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$$f^*_{T,R}(\xi,\eta) = \left\{\alpha\lambda\left[\bar{F}^*_p(\xi+\lambda+a) - a\bar{F}^{*\prime}_p(\xi+\lambda+a)\right] + \frac{\beta\vartheta^3}{2}\left[\bar{F}^{*\prime\prime}_p(\xi+\vartheta+a) - a\bar{F}^{*\prime\prime}_p(\xi+\vartheta+a)\right]\right\}$$
$$r^*_k(\eta) + \{-\alpha a^2\bar{F}^{*\prime}_p(\xi+\lambda+a)$$
$$+\beta a^2\left[-\bar{F}^{*\prime}_p(\xi+\vartheta+a) + \vartheta\bar{F}^{*\prime\prime}_p(\xi+\vartheta+a) - \frac{1}{2}\vartheta^2\bar{F}^{*\prime\prime\prime}_p(\xi+\vartheta+a)\right]\Bigg] r^*_h(\eta)$$
$$+\left(\alpha\left[f^*_p(\xi+\lambda+a) - af^{*\prime}_p(\xi+\lambda+a)\right] + \beta\left[f^*_p(\xi+\vartheta+a) - (\vartheta+a)f^{*\prime}_p(\xi+\vartheta+a)\right.\right.$$
$$+\left(\frac{\vartheta(\vartheta+2a)}{2}\right)\bar{f}^{*\prime\prime}_p(\xi+\lambda+a) - \frac{1}{2}a\vartheta^2\bar{f}^{*\prime\prime\prime}_p(\xi+\lambda+a)\Big]\Bigg\} r^*_p(\eta) \quad (15)$$

Using (10), (11), (12), (13) and(14) we rewrite (15) in terms of the transform of survivor function and its derivatives.We get

$$f^*_{T,R}(\xi,\eta) = \quad \{\alpha\lambda\left[\bar{F}^*_p(\xi+\lambda+a) - a\bar{F}^{*\prime}_p(\xi+\lambda+a)\right]$$
$$+\frac{\beta\vartheta^3}{2}\left[\bar{F}^{*\prime\prime}_p(\xi+\vartheta+a) - a\bar{F}^{*\prime\prime\prime}_p(\xi+\vartheta+a)\right]\Big\} r^*_k(\eta) + \{-\alpha a^2\bar{F}^{*\prime}_p(\xi+\lambda+a)$$
$$+\beta a^2\left[-\bar{F}^{*\prime}_p(\xi+\vartheta+a) + \vartheta\bar{F}^{*\prime\prime}_p(\xi+\vartheta+a) - \frac{1}{2}\vartheta^2\bar{F}^{*\prime\prime\prime}_p(\xi+\vartheta+a)\right]\Big\} r^*_h(\eta)$$
$$+\{\alpha\left[1-(\xi+\lambda)\bar{F}^*_p(\xi+\lambda+a) + a(\xi+\lambda+a)\bar{F}^{*\prime}_p(\xi+\lambda+a)\right]$$
$$+\beta\big(1-\xi\bar{F}^*_p(\xi+\vartheta+a) + \left((\vartheta+a)\xi+a^2\right)\bar{F}^{*\prime}_p(\xi+\vartheta+a)$$
$$-\left(\left(\frac{(\vartheta(\vartheta+2a)}{2}\right)\xi + \frac{1}{2}\vartheta(\vartheta+2a^2)\right)\bar{F}_p^{*\prime\prime\prime}(\xi+\vartheta+a)$$
$$+\frac{1}{2}a\vartheta^2(\xi+\vartheta+a)\bar{F}_p^{*\prime\prime\prime}(\xi+\vartheta+a))\} r^*_p(\eta) \quad (16)$$

We can verify the above satisfies $f^*_{T,R}(0,0) = 1$ as follows. Taking $\xi = 0 = \eta$ in (16) and noting $r^*_\#(0) = 1$ we get

$$f^*_{T,R}(0,0) = \left\{\alpha\lambda\left[\bar{F}^*_p(\lambda+a) - a\bar{F}^{*\prime}_p(\lambda+a)\right] + \frac{\beta\vartheta^3}{2}\left[\bar{F}^{*\prime\prime}_p(\vartheta+a) - a\bar{F}^{*\prime\prime\prime}_p(\vartheta+a)\right]\right\}$$
$$+\left\{-\alpha a^2\bar{F}^{*\prime}_p(\lambda+a) + \beta a^2\left[-\bar{F}^{*\prime}_p(\vartheta+a) + \vartheta\bar{F}^{*\prime\prime}_p(\vartheta+a) - \frac{1}{2}\vartheta^2\bar{F}^{*\prime\prime\prime}_p(\vartheta+a)\right]\right\}$$
$$+\left\{\alpha\left[(1-\lambda)\bar{F}^*_p(\lambda+a) + a(\lambda+a)\bar{F}^{*\prime}_p(\lambda+a)\right] + \beta\left(\left(1 + a^2\bar{F}^{*\prime}_p(\vartheta+a)\right)\right.\right.$$
$$\left.\left.-\left(\frac{1}{2}\vartheta(\vartheta^2+2a^2)\right)\bar{F}^{*\prime\prime}_p(\vartheta+a) + \frac{1}{2}a\vartheta^2(\vartheta+a)\bar{F}^{*\prime\prime\prime}_p(\vartheta+a)\right)\right\}$$
$$=\alpha+\beta$$
$$=1$$

since the coefficients of $\bar{F}^*_p$ and its derivatives add to zero. The Laplace transforms of Marginal pdfs of T and R are as follows.

$$f^*_T(\xi) = f^*_{\tau,R}(\xi,0)$$
$$= 1 - \xi\left[\alpha\bar{F}^*_p(\xi+\lambda+a) + \beta\bar{F}^*_p(\xi+\vartheta+a) - \alpha a\bar{F}^{*\prime}_p(\xi+\lambda+a)\right.$$
$$\left.-\beta(\vartheta+a)\bar{F}^{*\prime}_p(\xi+\vartheta+a) + \beta\left(\frac{\vartheta(\vartheta+2a)}{2}\right)\bar{F}^{*\prime\prime}_p(\xi+\vartheta+a) - \beta\frac{1}{2}a\theta^2\,\bar{F}_p^{*\prime\prime\prime(\xi+\vartheta+a)}\right] \quad (17)$$

We may note

$$E(T) = -\frac{\partial}{\partial\xi}f^*_{T,R}(\xi,\eta)_{(0,0)}$$
$$= -\frac{\partial}{\partial\xi}f^*_T(\xi)_{\xi=0}$$
$$= \alpha\bar{F}^*_p(\lambda+a) + \beta\bar{F}^*_p(\vartheta+a) - \alpha a\bar{F}^{*\prime}_p(\lambda+a) - \beta(\vartheta+a)\bar{F}^{*\prime}_p(\vartheta+a)$$

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$$+\beta\left(\frac{\vartheta(\vartheta+2a)}{2}\right)\bar{F}_p^{*''}(\vartheta+a)-\beta\frac{1}{2}a\vartheta^2\bar{F}_p^{*'''}(\vartheta+a). \tag{18}$$

The second moment of T, using (17), $E(T^2)=\frac{\partial^2}{\partial\xi^2}f_{T,R}^*(\xi,\eta)_{(0,0)}=\frac{\partial^2}{\partial\xi^2}f_T^*(\xi)_{\xi=0}$

$$E(T^2)=-2\alpha\bar{F}_p^{*\prime}(\lambda+a)-2\beta\bar{F}_p^{*\prime}(\vartheta+a)+2\alpha a\bar{F}_p^{*\prime\prime}(\lambda+a)$$
$$+2(\vartheta+a)\beta\bar{F}_p^{*''}(\vartheta+a)-\beta\vartheta(\vartheta+2a)\bar{F}_p^{*'''}(\vartheta+a)+\beta a\vartheta^2\bar{F}_p^{*''''}(\vartheta+a). \tag{19}$$

$$\begin{aligned}f_R^*(\eta)=&f_{T,R}^*(0,\eta)\\=&\left\{\alpha\lambda\left[\bar{F}_p^*(\lambda+a)-a\bar{F}_p^{*\prime}(\lambda+a)\right]+\frac{\beta\vartheta^3}{2}\left[\bar{F}_p^{*\prime\prime}(\vartheta+a)-a\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\right]\right\}r_k^*(\eta)\\&+\left\{-\alpha a^2\bar{F}_p^{*'}(\lambda+a)+\beta a^2\left[-\bar{F}_p^{*\prime}(\vartheta+a)+\vartheta\bar{F}_p^{*\prime\prime}(\vartheta+a)-\frac{1}{2}\vartheta^2\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\right]\right\}r_h^*(\eta)\\&+\{\alpha\left[1-\lambda\bar{F}_p^*(\lambda+a)+a(\lambda+a)\bar{F}_p^{*'}(\lambda+a)\right]+\beta\big(1-a^2\bar{F}_p^{*'}(\vartheta+a)\\&-\frac{1}{2}\vartheta(\vartheta^2+2a^2)\bar{F}_p^{*\prime\prime}(\vartheta+a)+\frac{1}{2}a\vartheta^2(\vartheta+a)\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\big)\}r_p^*(\eta).\end{aligned}$$

Similarly, we can find $E(R)$ and $E(R^2)$

$$\begin{aligned}E(R)=&-\frac{\partial}{\partial\eta}f_{T,R}^*(\xi,\eta)_{(0,0)}\\=&-\frac{\partial}{\partial\eta}f_R^*(\eta)_{\eta=0}\\=&\left\{\alpha\lambda\left[\bar{F}_p^*(\lambda+a)-a\bar{F}_p^{*'}(\lambda+a)\right]+\frac{\beta\vartheta^3}{2}\left[\bar{F}_p^{*''}(\vartheta+)-a\bar{F}_p^{*'''}(\vartheta+a)\right]\right\}E(R_k)\\&+\left\{-\alpha a^2\bar{F}_p^{*'}(\lambda+a)+\beta a^2\left[-\bar{F}_p^{*'}(\vartheta+a)+\vartheta\bar{F}_p^{*''}(\vartheta+a)-\frac{1}{2}\vartheta^2\bar{F}_p^{*'''}(\vartheta+a)\right]\right\}E(R_h)\\&+\left\{\alpha\left[1-\lambda\bar{F}_p^*(\lambda+a)+a(\lambda+a)\bar{F}_p^{\phi'}(\lambda+a)\right]+\beta\big(1-a^2\bar{F}_p^{*'}(\vartheta+a)\right.\\&-\frac{1}{2}\vartheta(\vartheta+2a^2)\bar{F_p}^{*''}(\vartheta+a)+\frac{1}{2}a\vartheta^2(\vartheta+a)\bar{F_p}^{*'''}(\vartheta+a)\}E(R_p)\end{aligned} \tag{20}$$

Similarly, the second moment of R is

$$E(R^2)=\frac{\partial^2}{\partial\eta^2}f_{\tau,R}^*(\xi,\eta)_{(0,0)}=\frac{\partial^2}{\partial\eta^2}f_R^*(\eta)_{\eta=0}$$

Using (20) we note

$$\begin{aligned}E(R^2)=&\left\{\alpha\lambda\left[\bar{F}_p^*(\lambda+a)-a\bar{F}_p^{*'}(\lambda+a)\right]+\frac{\beta\vartheta^3}{2}\left[\bar{F}_p^{*''}(\vartheta+a)-a\bar{F}_p^{*''}(\vartheta+a)\right]\right\}E(R_k^2)\\&+\left\{-\alpha a^2\bar{F}_p^{*'}(\lambda+a)+\beta a^2\left[-\bar{F}_p^{*'}(\vartheta+a)+\vartheta\bar{F}_p^{*''}(\vartheta+a)-\frac{1}{2}\vartheta^2\bar{F}_p^{*'''}(\vartheta+a)\right]\right\}E(R_h^2)\\&+\{\alpha\left[1-\lambda\bar{F}_p^*(\lambda+a)+a(\lambda+a)\bar{F}_p^{*'}(\lambda+a)\right]+\beta\big(1-a^2\bar{F}_p^{*'}(\vartheta+a)\\&-\frac{1}{2}\vartheta(\vartheta+2a^2)\bar{F_p}^{*''}(\vartheta+a)+\frac{1}{2}a\vartheta^2(\vartheta+a)\bar{F_p}^{*'''}(\vartheta+a)\}E(R_p^2)\big\}\end{aligned} \tag{21}$$

Section 2: Model 2 with AkashShanker Mixture of E(ϑ) and E$(3,\vartheta)$ for Kidney Ailment time for treatment

The Cdf of Akashshanker (AS) distribution is a mixture of Exponential Cdf with parameter ϑ and Gamma Cdf with phase 3 and parameter ϑ. It can be seen the Cdf of AS distribution as

$$F_k(x)=\left(\frac{\vartheta^2}{\vartheta^2+2}\right)\left(1-e^{-\vartheta x}\right)+\left(\frac{2}{\vartheta^2+2}\right)\left(1-e^{-\vartheta x}-\vartheta xe^{-\vartheta x}-\frac{1}{2}\vartheta^2x^2e^{-\vartheta x}\right)$$

$$= 1 - \left[1 + \frac{\vartheta x(\vartheta x + 2)}{\vartheta^2 + 2}\right] e^{-\vartheta x}. \tag{22}$$

The pdf of AS can be seen from (22) as

$$f_k(x) = \left(\frac{\vartheta^3}{\vartheta^2 + 2}\right)(1 + x^2)e^{-\vartheta x} \tag{23}$$

Now (23) can be written as

$$f_k(x) = \left(\frac{\vartheta^2}{\vartheta^2 + 2}\right)\left(\vartheta e^{-\vartheta x}\right) + \left(\frac{2}{\vartheta^2 + 2}\right)\left(\vartheta^3 \frac{x^2}{2} e^{-\vartheta x}\right). \tag{24}$$

We note the mixture probabilities

$$\alpha = \left(\frac{\vartheta^2}{\vartheta^2 + 2}\right) \text{ and } \beta = \left(\frac{2}{\vartheta^2 + 2}\right) \text{ where } \alpha + \beta = 1. \tag{25}$$

So, the Laplace Transform (LT) of pdf AS is

$$f_k^*(s) = \left(\frac{\vartheta^2}{\vartheta^2 + 2}\right)\left(\frac{\vartheta}{\vartheta + s}\right) + \left(\frac{2}{\vartheta^2 + 2}\right)\left(\frac{\vartheta}{\vartheta + s}\right)^3. \tag{26}$$

So, we get

$$f_k^*(s) = \alpha\left(\frac{\vartheta}{\vartheta + s}\right) + \beta\left(\frac{\vartheta}{\vartheta + s}\right)^3. \tag{27}$$

where α and β are given by (25).
We notefrom (22), the random variable X of AS is

$$X = \begin{cases} U \text{ with probability } \alpha = \left(\frac{\vartheta^2}{\vartheta^2 + 2}\right) \\ V \text{ with probability } \beta = \left(\frac{2}{\vartheta^2 + 2}\right) \end{cases} \tag{28}$$

where U has Exponential distribution with parameter ϑ and V has Gamma distribution with shape 3 and scale ϑ. Here the random variable X is not a single random variable but it is a mixture of Exponential (ϑ) and Gamma ($3, \vartheta$) with probabilities of (25).

Assumtions:

(i) Kidney of a patient gets affected at a random time followingAS distribution whose Cdf and pdf are given by (22) and (23).
(ii) (ii) Heart of the patient gets affected in a random time following Erlang distribution with phase 2 and with scale parameter a whose pdf is

$$f_h(x) = a^2 x e^{-ax} \text{ and Cdf} F_h(x) = 1 - e^{-ax} - \text{ axe } e^{-ax} \text{ where } x, a > 0.$$

(iii) The patient is also provided prophylactic treatment at a random time following general distribution with pdf $f_p(x)$ and with Cdf$F_p(x)$.
(iv) At time 0 there is no damage to any organ and the random times described above are independent of each other.
(v) The patient is referred for treatment upon whichever event happens first: organ failure or prophylactic purpose.
(vi) The treatment time for Kidney, Heart or for prophylactic purpose are independent with pdf and Cdf are respectively $(r_k(y), R_k(y)), (r_h(y), R_h(y))$ and $\left(r_p(y), R_p(y)\right)$.

Analysis:

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Considering the three failures and three treatments we note the joint pdf of T, the time to send for treatment to hospital and R, the hospital treatment time for the patient is

$$f_{T,R}(x,y) = f_k(x)\bar{F}_h(x)\bar{F}_p(x)r_k(y) + \bar{F}_k(x)f_h(x)\bar{F}_p(x)r_h(y) + \bar{F}_k(x)\bar{F}_h(x)f_p(x)r_p(y) \quad (29)$$

where $\bar{G}_L(x) = 1 - G_L(x)$ where $G_L(x)$ is any distribution function. Using equations (22) to (29) we note the joint pdf is

$$\begin{aligned} f_{T,R}(x,y) =& \left(\frac{\vartheta^3}{\vartheta^2+2}\right)(1+x^2)e^{-\vartheta x}(e^{-ax}+axe^{-ax})\bar{F}_p(x)r_k(y) \\ &+\left[1+\frac{\vartheta x(\vartheta x+2)}{\vartheta^2+2}\right]e^{-\vartheta x}a^2xe^{-ax}\bar{F}_p(x)r_h(y) \\ &+\left[1+\frac{\vartheta x(\vartheta x+2)}{\vartheta^2+2}\right]e^{-\vartheta x}(e^{-ax}+axe^{-ax})f_p(x)r_p(y). \end{aligned} \quad (30)$$

Laplace transform of thejoint pdf of (T, R) is

$$\begin{aligned} & f_{T,R}^*(\xi,\eta) = \iint_0^\infty e^{-\xi x}e^{-\eta y}f_{T,R}(x,y)dxdy \\ &= \int_0^\infty \int_0^\infty e^{-\xi x}e^{-\eta y}\left[\left(\frac{\vartheta^3}{\vartheta^2+2}\right)(1+x^2)e^{-\vartheta x}(e^{-ax}+axe^{-ax})\bar{F}_p(x)r_k(y)\right. \\ &+\left[1+\frac{\vartheta x(\vartheta x+2)}{\vartheta^2+2}\right]e^{-\vartheta x}a^2xe^{-ax}\bar{F}_p(x)r_h(y) \\ &\qquad +\left[1+\frac{\vartheta x(\vartheta x+2)}{\vartheta^2+2}\right]e^{-\vartheta x}(e^{-ax}+axe^{-ax})f_p(x)r_p(y)]\text{dxdy} \end{aligned} \quad (31)$$

Results have been derived for a general mixture in Model 1. Substituting $\lambda = \vartheta$ and $\alpha = \left(\frac{\vartheta^2}{\vartheta^2+2}\right)$ and $\beta = \left(\frac{2}{\vartheta^2+2}\right)$ it can be seen from (16) that

$$\begin{aligned} f_{T_{,R}^*(\xi,\eta)} =& \left\{\left(\frac{\vartheta^2}{\vartheta^2+2}\right)\vartheta\left[\bar{F}_p^*(\xi+\vartheta+a) - a\bar{F}_p^{*\prime}(\xi+\vartheta+a)\right]\right. \\ &\left.+\left(\frac{2}{\vartheta^2+2}\right)\frac{\vartheta^3}{2}\left[\bar{F}_p^{*\prime\prime}(\xi+\vartheta+a) - a\bar{F}_p^{*\prime\prime\prime}(\xi+\vartheta+a)\right]\right\}r_k^*(\eta) \\ &+\left\{-\left(\frac{\vartheta^2}{\vartheta^2+2}\right)a^2\bar{F}_p^{*\prime}(\xi+\vartheta+a)\right. \\ &\left.+\left(\frac{2}{\vartheta^2+2}\right)a^2\left[-\bar{F}_p^{*\prime\prime}(\xi+\vartheta+a)+\vartheta\bar{F}_p^{*\prime\prime}(\xi+\vartheta+a)-\frac{1}{2}\vartheta^2\bar{F}_p^{*\prime\prime\prime}(\xi+\vartheta+a)\right]\right\}r_h^*(\eta) \\ &+\left\{\left(\frac{\vartheta^2}{\vartheta^2+2}\right)\left[1-(\xi+\vartheta)\bar{F}_p^*(\xi+\vartheta+a)+a(\xi+\vartheta+a)\bar{F}_p^{*\prime}(\xi+\vartheta+a)\right]\right. \\ &+\left(\frac{2}{\vartheta^2+2}\right)\left(1-\xi\bar{F}_p^*(\xi+\vartheta+a)+((\vartheta+a)\xi+a^2)\bar{F}_p^{*\prime}(\xi+\vartheta\right. \\ &+a))-\left(\left(\frac{\vartheta(\vartheta+2a)}{2}\right)\xi+\frac{1}{2}\vartheta(\vartheta^2+2a^2)\right)\bar{F}_p^{*\prime\prime}(\xi+\vartheta+a) \\ &\left.\left.+\frac{1}{2}a\vartheta^2(\xi+\vartheta+a)\bar{F}_p^{*\prime\prime\prime}(\xi+\vartheta+a)\right)\right\}r_p^*(n). \end{aligned}$$

This reduces to

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$$f_{T,R}^{*}(\xi,\eta) = \left(\frac{\vartheta^3}{\vartheta^2+2}\right)\{\bar{F}_p^{*}(\xi+\vartheta+a) - a\bar{F}_p^{*'}(\xi+\vartheta+a) + \bar{F}_p^{*''}(\xi+\vartheta+a) - a\bar{F}_p^{*'''}(\xi+\vartheta+a)\}r_k^{*}(\eta) + \left\{-a^2\bar{F}_p^{*'}(\xi+\vartheta+a) + \left(\frac{a^2}{\vartheta^2+2}\right)\left[2\vartheta\bar{F}_p^{*''}(\xi+\vartheta+a) - \vartheta^2\bar{F}_p^{*'''}(\xi+\vartheta+a)\right]\right\}r_h^{*}(\eta) + \left\{1 - \left(\xi + \frac{\vartheta^3}{\vartheta^2+2}\right)\bar{F}_p^{*'}(\xi+\vartheta+a) + \left[\left(a + \frac{2\vartheta}{\vartheta^2+2}\right)\xi + \frac{a}{\vartheta^2+2}(\vartheta^3+\vartheta^2 a+2a)\right]\bar{F}_p^{*}(\xi+\vartheta+a) + \left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2(\xi+\vartheta+a)\bar{F}_p^{*'''}(\xi+\vartheta+a)\right\}r_p^{*}(\eta) \quad (32)$$

The Laplace transform of the marginal pdf of T is

$$f_{T,R}^{*}(\xi,0) = f_T^{*}(\xi) = 1 - \xi\left[\bar{F}_p^{\ \ *}(\xi+\vartheta+a) - \left(a + \frac{2\vartheta}{\vartheta^2+2}\right)\bar{F}_p^{*'}(\xi+\vartheta+a) + \left(\frac{1}{\vartheta^2+2}\right)\vartheta(\vartheta+2a)\bar{F}_p^{*''}(\xi+\vartheta+a) - \left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2\bar{F}_p^{*'''}(\xi+\vartheta+a)\right].$$

We now using (18) write down $E(T)$ below.

$$E(T) = \alpha\bar{F}_p^{*}(\lambda+a) + \beta\bar{F}_p^{*}(\vartheta+a) - (a+\beta\vartheta)\bar{F}_p^{*'}(\vartheta+a) + \beta\frac{\vartheta(\vartheta+2a)}{2}\bar{F}_p^{*''}(\vartheta+a) - \beta\frac{1}{2}a\vartheta^2\bar{F}_p^{*'''}(\vartheta+a)$$

This on substituting

$$\lambda = \vartheta, \alpha = \left(\frac{\vartheta^2}{\vartheta^2+2}\right) \text{and} \beta = \left(\frac{2}{\vartheta^2+2}\right)$$

becomes

$$E(T) = \bar{F}_p^{*}(\vartheta+a) - \left(a + \frac{2\vartheta}{\vartheta^2+2}\right)\bar{F}_p^{*'}(\vartheta+a) + \left(\frac{\vartheta(\vartheta+2a)}{\vartheta^2+2}\right)\bar{F}_p^{*''}(\vartheta+a) - \bar{F}_p^{'''}(\vartheta+a)\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2. \quad (33)$$

We now using (19) write down $E(T^2)$ below.

$$E(T^2) = -2\alpha\bar{F}_p^{*'}(\lambda+a) - 2\beta\bar{F}_p^{*'}(\vartheta+a) + 2\alpha a\bar{F}_p^{*''}(\lambda+a) + 2(\vartheta+a)\beta\bar{F}_p^{''}(\vartheta+a) - \beta\vartheta(\vartheta+2a)\bar{F}_p^{*'''}(\vartheta+a) + \beta a\vartheta^2\bar{F}_p^{*''''}(\vartheta+a).$$

This on substituting

$$\lambda = \vartheta, \alpha = \left(\frac{\vartheta^2}{\vartheta^2+2}\right) \text{and } \beta = \left(\frac{2}{\vartheta^2+2}\right)$$

becomes

$$E(T^2) = -2\bar{F}_p^{*'}(\vartheta+a) + 2\left(a + \frac{2\vartheta}{\vartheta^2+2}\right)\bar{F}_p^{*''}(\vartheta+a) - 2\left(\frac{\vartheta(\vartheta+2a)}{\vartheta^2+2}\right)\bar{F}_p^{*''}(\vartheta+a) + 2\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2\bar{F}_p^{*''''}(\vartheta+a). \quad (34)$$

So,

$$\text{Var}(T) = E(T^2) - E(T)^2 \quad (35)$$

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gives the variance.

$$\begin{aligned}f_R^*(\eta) =&f_{T,R}^*(0,\eta)\\ =&\left(\frac{\vartheta^3}{\vartheta^2+2}\right)\{\bar{F}_p^*(\vartheta+a) - a\bar{F}_p^{*\prime}(\vartheta+a) + \bar{F}_p^{*\prime\prime}(\vartheta+a) - a\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\}r_k^*(\eta)\\ &+\left\{-a^2\bar{F}_p^{*\prime}(\vartheta+a) + \left(\frac{a^2}{\vartheta^2+2}\right)\left[2\vartheta\bar{F}_p^{*\prime\prime}(\vartheta+a) - \vartheta^2\bar{F}_p^{*\prime\prime}(\vartheta+a)\right]\right\}r_h^*(\eta)\\ &+\left\{1-\left(\frac{\vartheta^3}{\vartheta^2+2}\right)\right)\bar{F}_p^{*\prime}(\vartheta+a)\\ &+\left[\frac{a}{\vartheta^2+2}(\vartheta^3+\vartheta^2a+2a)\right]\bar{F}_p^*(\vartheta+a) + \left(\frac{1}{\vartheta^2+2}\right)[-\vartheta^3-2a^2\vartheta]\bar{F}_p^{*\prime\prime}(\vartheta+a)\\ &+\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2(\vartheta+a)\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\right\}r_p^*(\eta).\end{aligned}$$

Using the above analysis and (32), we note

$$\begin{aligned}E(R) =&-\frac{\partial}{\partial\eta}f_{T,R}^*(\xi,\eta)_{(0,0)}\\ =&\left(\frac{\vartheta^3}{\vartheta^2+2}\right)\{\bar{F}_p^*(\vartheta+a) - a\bar{F}_p^{*\prime}(\vartheta+a) + \bar{F}_p^{*\prime\prime}(\vartheta+a) - a\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\}E(R_k)\\ &+\left\{-a^2\bar{F}_p^{*\prime}(\vartheta+a) + \left(\frac{a^2}{\vartheta^2+2}\right)\left[2\vartheta\bar{F}_p^{*\prime\prime}(\vartheta+a) - \vartheta^2\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\right]\right\}E(R_h)\\ &+\left\{1-\left(\frac{\vartheta^3}{\vartheta^2+2}\right)\right)\bar{F}_p^*(\vartheta+a)\\ &+\left[\frac{a}{\vartheta^2+2}(\vartheta^3+\vartheta^2a+2a)\right]\bar{F}_p^{*\prime}(\vartheta+a) + \left(\frac{1}{\vartheta^2+2}\right)[-\vartheta^3-2a^2\vartheta]\bar{F}_p^{*\prime\prime}(\vartheta+a)\end{aligned} \tag{36}$$

$$+\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2(\vartheta+a)\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\}\mathrm{E}(\mathrm{R_p})$$

Using the above analysis and (36), we note

$$\begin{aligned}E(R^2) =&\frac{\partial^2}{\partial\eta^2}f_{T,R}^*(\xi,\eta)_{(0,0)}\\ =&\left(\frac{\vartheta^3}{\vartheta^2+2}\right)\{\bar{F}_p^*(\vartheta+a) - a\bar{F}_p^{*\prime}(\vartheta+a) + \bar{F}_p^{*\prime\prime}(\vartheta+a) - a\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\}E(R_k^2)\\ &+\left\{-a^2\bar{F}_p^{*\prime}(\vartheta+a) + \left(\frac{a^2}{\vartheta^2+2}\right)\left[2\vartheta\bar{F}_p^{*\prime\prime}(\vartheta+a) - \vartheta^2\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\right]\right\}E(R_h^2)\\ &+\left\{1-\left(\frac{\vartheta^3}{\vartheta^2+2}\right)\right)\bar{F}_p^*(\vartheta+a)\\ &+\left[\frac{a}{\vartheta^2+2}(\vartheta^3+\vartheta^2a+2a)\right]\bar{F}_p^{*\prime}(\vartheta+a) + \left(\frac{1}{\vartheta^2+2}\right)[-\vartheta^3-2a^2\vartheta]\bar{F}_p^{*\prime\prime}(\vartheta+a)\end{aligned} \tag{37}$$

$$+\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2(\vartheta+a)\bar{F}_p^{*\prime\prime\prime}(\vartheta+a)\}E(R_p^2)$$

$$\mathrm{Var}(R) = E(R^2) - E(R)^2 \tag{38}$$

gives the variance.

Section 3: Model 1 with Mixture of $E(\lambda)$ and $E(3,\vartheta)$ for Kidney Ailment time for treatment with Exponential Prophylactic time

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In this section we study again Model 1 where the general mixture assumption of Exponential and Erlang 3 distribution for Kidney ailment time is considered. Keeping all assumptions of Model 1 except assumption (III) in tact we assume that the Prophylactic arrival rate is constant b. This means that the Prophylactic timedistribution is exponential with parameter b. It is also somewhat reasonable to assume the exponential distribution for Prophylactic time since it has wide applications in model building in Operations Research and Queueing theory where constant arrival rate is frequently considered in models. We list the assumptions of the model below.

Assumptions:

(I) Kidney gets affected at a random time following probability distribution which is a mixture of exponential distribution with parameter λ with probability α and Erlang distribution with phase 3 and parameter ϑ with probability $\beta = 1 - \alpha$. This mixture hasCdf is

$$F_k(x) = \alpha\left(1 - e^{-\lambda x}\right) + \beta\left(1 - e^{-\vartheta x} - \vartheta x e^{-\vartheta x} - \frac{1}{2}\vartheta^2 x^2 e^{-\vartheta x}\right)$$

with pdf $f_k(x) = \alpha\left(\lambda e^{-\lambda x}\right) + \beta\left(\vartheta^3 \frac{x^2}{2} e^{-\vartheta x}\right)$.

(II) Heart gets affected in a random time following Erlang distribution with phase 2 and with scale parameter a whose pdf is $f_h(x) = \frac{a^2 x e^{-ax}}{1} = a^2 x e^{-ax}$ with Cdf$F_h(x) = 1 - e^{-ax} - axe^{-ax}$ where $x, a > 0$.

(III) The patient also receives prophylactic treatment at a random time, following an exponential distribution with parameter b . This means that the Cumulative Distribution Function (CDF) and Probability Density Function (PDF) are, respectively, $F_p(x) = 1 - e^{-bx}; f_p(x) = be^{-bx}$.

(IV) At time 0 there is no damage to any organ and the random times described above are independent of each other.

(V) The patient is referred for treatment upon the earlier of either organ failure or for prophylactic purposes.

(VI) The treatment time for Kidney, Heart or for Prophylactic purpose are independent with pdf and Cdf which are respectively $(r_k(y), R_k(y)), (r_h(y), R_h(y))$ and $\left(r_p(y), R_p(y)\right)$.

Analysis:

We now consider here a special case in which the Prophylactic time has exponential distribution with parameter b . We note the Cdf is $F_p(x) = 1 - e^{-bx}$ with pdf $f_p(x) = be^{-bx}$. Note the Laplace transform of the survivor function and other functions are as follows.

$$\bar{F}_p^*(s) = \frac{1}{b+s};$$

$$f_b^*(s) = \frac{b}{b+s};$$

$$\bar{F}_p^{*\prime}(s) = \frac{-1}{(b+s)^2};$$

$$\bar{F}_p^{*\prime\prime}(s) = \frac{2}{(b+s)^3};$$

$$\bar{F}_p^{*\prime\prime}(s) = \frac{-6}{(b+s)^4};$$

$$\bar{F}_p^{*\prime\prime\prime\prime}(s) = \frac{24}{(b+s)^5};$$

$$\bar{F}_p^{*\prime\prime\prime\prime\prime}(s) = \frac{-120}{(b+s)^6};$$

$$f_b^{*\prime}(s) = \frac{-b}{(b+s)^2};$$

$$f_b^{*\prime\prime}(s) = \frac{2b}{(b+s)^3};$$

and

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$$f_b^{*'''}(s) = \frac{-6b}{(b+s)^4}. \tag{39}$$

Considering the three failures and three treatments we note the joint pdf of T, the time to send for treatment to hospital and R, the hospital treatment time for the patient is

$$f_{T,R}(x,y) = f_k(x)\bar{F}_h(x)\bar{F}_p(x)r_k(y) + \bar{F}_k(x)f_h(x)\bar{F}_p(x)r_h(y) + \bar{F}_k(x)\bar{F}_h(x)f_p(x)r_p(y).$$

Using Model 1 results we get from (17), the Joint Laplace transform of the pdf is

$$\begin{aligned} f_{T,R}^*(\xi,\eta) = \ & \{\alpha\lambda[\bar{F}_p^*(\xi+\lambda+a) - a\bar{F}_p^{*'}(\xi+\lambda+a)] \\ & +\frac{\beta\vartheta^3}{2}\left[\bar{F}_p^{*''}(\xi+\vartheta+a) - a\bar{F}_p^{*'''}(\xi+\vartheta+a)\right]\Big\} r_k^*(\eta) + \{-\alpha a^2\bar{F}_p^{*'}(\xi+\lambda+a) \\ & +\beta a^2\left[-\bar{F}_p^{*'}(\xi+\vartheta+a) + \vartheta\bar{F}_p^{*''}(\xi+\vartheta+a) - \frac{1}{2}\vartheta^2\bar{F}_p^{*'''}(\xi+\vartheta+a)\right]\Big] r_h^*(\eta) \\ + \ & \{\alpha[1-(\xi+\lambda)\bar{F}_p^*(\xi+\lambda+a) + a(\xi+\lambda+a)\bar{F}_p^{*'}(\xi+\lambda+a)] \\ + \ & \beta\big(1-\xi\bar{F}_p^*(\xi+\vartheta+a) + ((\vartheta+a)\xi+a^2)\bar{F}_p^{*'}(\xi+\vartheta+a) \\ - & \left(\left(\frac{(\vartheta(\vartheta+2a)}{2}\right)\xi + \frac{1}{2}\vartheta(\vartheta^2+2a^2)\right)\bar{F}_p^{*''(\xi+\vartheta+a)} \\ +\frac{1}{2} & a\vartheta^2(\xi+\vartheta+a)\bar{F}_p^{*'''}(\xi+\vartheta+a))\} r_p^*(\eta) \end{aligned}$$

Using (43) we can write the result for this model as follows.

$$\begin{aligned} f_{T,R}^*(\xi,\eta) = & \Big\{\alpha\lambda\left[\frac{1}{b+\xi+\lambda+a} + a\frac{1}{(b+\xi+\lambda+a)^2}\right] \\ & +\frac{\beta\vartheta^3}{2}\left[\frac{2}{(b+\xi+\vartheta+a)^3} + a\frac{6}{(b+\xi+\vartheta+a)^4}\right]\Big\} r_k^*(n) \\ + & \Big\{\alpha a^2\frac{1}{(b+\xi+\lambda+a)^2} + \beta a^2\left[\frac{1}{(b+\xi+\vartheta+a)^2} + \vartheta\frac{2}{(b+\xi+\vartheta+a)^3} + \frac{1}{2}\vartheta^2\frac{6}{(b+\xi+\vartheta+a)^4}\right] r_h^*(n) \\ & +\Big\{\alpha\left[1-(\xi+\lambda)\frac{1}{b+\xi+\lambda+a} - a(\xi+\lambda+a)\frac{1}{(b+\xi+\lambda+a)^2}\right] \\ & +\beta\left(1-\xi\frac{1}{b+\xi+\vartheta+a} - ((\vartheta+a)\xi+a^2)\frac{1}{(b+\xi+\vartheta+a)^2} - \left(\left(\frac{\vartheta(\vartheta+2a)}{2}\right)\xi\right.\right. \\ & \left.\left.+\frac{1}{2}\vartheta(\vartheta^2+2a^2)\right)\frac{2}{(b+\xi+\vartheta+a)^3}\frac{1}{2}a\vartheta^2(\xi+\vartheta+a)\frac{6}{(b+\xi+\vartheta+a)^4}\right)\Big\} r_p^*(\eta). \end{aligned} \tag{40}$$

The Laplace transform of marginal pdf of T and r can be written as follows model 1 equation (17).

$$\begin{aligned} f_T^*(\xi) = & f_{T,R}^*(\xi,0) \\ & =1-\xi\left[\alpha\frac{1}{b+\xi+\lambda+a} + \beta\frac{1}{b+\xi+\vartheta+a} + \alpha a\frac{1}{(b+\xi+\lambda+a)^2}\right. \\ & +\beta(\vartheta+a)\frac{1}{(b+\xi+\vartheta+a)^2} + \beta\left(\frac{\vartheta(\vartheta+2a)}{2}\right)\frac{2}{(b+\xi+\vartheta+a)^3} \\ & \left.+\beta\tfrac{1}{2}a\vartheta^2\frac{6}{(b+\xi+\vartheta+a)^4}\right]. \end{aligned} \tag{41}$$

Using (18) and (40), $E(T)$ for this Model can be written for the Exponential Prophylactic case,

$$\begin{aligned} E(T) = & \alpha\frac{1}{b+\lambda+a} + \beta\frac{1}{b+\vartheta+a} + (a+\beta\vartheta)\frac{1}{(b+\vartheta+a)^2} \\ & +\beta\frac{1}{(b+\vartheta+a)^3}\vartheta(\vartheta+2a) + \beta a\vartheta^2\frac{3}{(b+\vartheta+a)^4}. \end{aligned} \tag{42}$$

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Using (19) and (40), $E(T^2)$ for this Model can be written for the Exponential Prophylactic case,

$$E(T^2) = 2\alpha \frac{1}{(b+\lambda+a)^2} + 2\beta \frac{1}{(b+\vartheta+a)^2} + 2\alpha a \frac{2}{(b+\lambda+a)^3} + 2(\vartheta+a)\beta \frac{2}{(b+\vartheta+a)^3} + \beta\vartheta(\vartheta+2a)\frac{6}{(b+\vartheta+a)^4} + \beta a\vartheta^2 \frac{24}{(b+\vartheta+a)^5}. \quad (43)$$

So,

$$\mathrm{Var}(T) = E(T^2) - E(T)^2 \text{ gives the variance.} \quad (44)$$

a	E(R)	Var(R)
0.2	0.5872	0.1952
0.4	0.5242	0.1944
0.6	0.4611	0.1856
0.8	0.3981	0.1689
1	0.3351	0.1443

Table 1: Value of E(T) and Var(T)

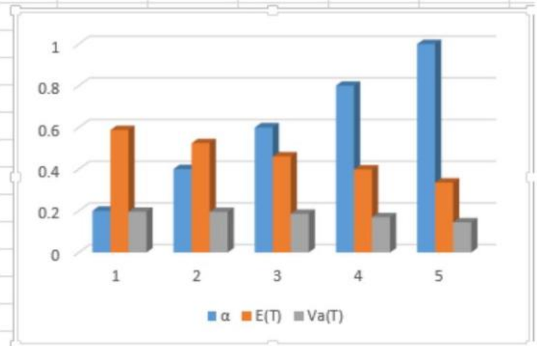

Figure 1: E(T) and Var(T) in Model 3

The Laplace transform of the Marginal pdf of R can be seen as follows from (40).

$$\begin{aligned} f_R^*(\eta) &= f_{T,R}^*(0,\eta) \\ &= \left\{\alpha\lambda\left[\frac{1}{b+\lambda+a} + a\frac{1}{(b+\lambda+a)^2}\right] + \frac{\beta\vartheta^3}{2}\left[\frac{2}{(b+\vartheta+a)^3} + a\frac{6}{(b+\vartheta+a)^4}\right]\right\} r_k^*(\eta) \\ &+ \left\{\alpha a^2 \frac{1}{(b+\lambda+a)^2}\right. \\ &\left. + \beta a^2\left[\frac{1}{(b+\vartheta+a)^2} + \vartheta\frac{2}{(b+\vartheta+a)^3} + \frac{1}{2}\vartheta^2\frac{6}{(b+\vartheta+a)^4}\right]\right\} r_h^*(n) \\ &+ \left\{\alpha\left[1 - \lambda\frac{1}{b+\lambda+a} - a(\lambda+a)\frac{1}{(b+\lambda+a)^2}\right] \cdot + \beta\left(1 - a^2\frac{1}{(b+\vartheta+a)^2}\right.\right. \\ &\left.\left. - \left(\frac{1}{2}\vartheta(\vartheta^2+2a^2)\right)\frac{2}{(b+\vartheta+a)^3} - \frac{1}{2}a\vartheta^2(\vartheta+a)\frac{6}{(b+\vartheta+a)^4}\right)\right\} r_p^*(\eta) \end{aligned}$$

We find

$$E(R) = -\frac{\partial}{\partial\eta} f_{T,R}^*(\xi,\eta)_{(0,0)}$$

$$\begin{aligned} E(R) &= \left\{\alpha\lambda\left[\frac{1}{b+\lambda+a} + a\frac{1}{(b+\lambda+a)^2}\right] + \frac{\beta\vartheta^3}{2}\left[\frac{2}{(b+\vartheta+a)^3} + a\frac{6}{(b+\vartheta+a)^4}\right]\right\} E(R_k) \\ &+ \left\{\alpha a^2\frac{1}{(b+\lambda+a)^2} + \beta a^2\left[\frac{1}{(b+\vartheta+a)^2} + \vartheta_{(b+\vartheta+a)^3} + \frac{1}{2}\vartheta^2\frac{6}{(b+\vartheta+a)^4}\right]\right\} E(R_h) \\ &+ \left\{\alpha\left[1 - \lambda\frac{1}{b+\lambda+a} - a(\lambda+a)\frac{1}{(b+\lambda+a)^2}\right]\right. \\ &+ \beta\left(1 - a^2\frac{1}{(b+\vartheta+a)^2}\left(\frac{1}{2}\vartheta(\vartheta^2+2a^2)\right)\frac{2}{(b+\vartheta+a)^3}\right. \\ &\left.\left. - \frac{1}{2}a\vartheta^2(\vartheta+a)\frac{6}{(b+\vartheta+a)^4}\right)\right\} \mathrm{E}(R_p) \end{aligned} \quad (45)$$

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Using the above analysis and (40), we find

$$E(R^2) = \frac{\partial^2}{\partial \eta^2} f^*_{T,R}(\xi,\eta)_{(0,0)}$$

$$\begin{aligned} E(R^2) =& \left\{\alpha\lambda\left[\frac{1}{b+\lambda+a} + a\frac{1}{(b+\lambda+a)^2}\right] + \frac{\beta\vartheta^3}{2}\left[\frac{2}{(b+\vartheta+a)^3} + a\frac{6}{(b+\vartheta+a)^4}\right]\right\} E(R_k^2) + \left\{\alpha a^2 \frac{1}{(b+\lambda+a)^2}\right. \\ & \left. +\beta a^2\left[\frac{1}{(b+\vartheta+a)^2} + \vartheta\frac{2}{(b+\vartheta+a)^3} + \frac{1}{2}\vartheta^2\frac{6}{(b+\vartheta+a)^4}\right]\right\} E(R_h^2) \\ & +\left\{\alpha\left[1-\lambda\frac{1}{b+\lambda+a} - a(\lambda+a)\frac{1}{(b+\lambda+a)^2}\right]\cdot +\beta\left(1-a^2\frac{1}{(b+\vartheta+a)^2}\right.\right. \\ & \left.\left.\left(\frac{1}{2}\vartheta(\vartheta^2+2a^2)\right)\frac{2}{(b+\vartheta+a)^3} - \frac{1}{2}a\vartheta^2(\vartheta+a)\frac{6}{(b+\vartheta+a)^4}\right)\right\} \mathrm{E}(R_p^2) \end{aligned} \quad (46)$$

$$\mathrm{Var}(R) = E(R^2) - E(R)^2 \quad (47)$$

gives the variance.

a	E(R)	Var(R)
0.2	0.1004	0.0401
0.4	0.1143	0.0441
0.6	0.1283	0.0477
0.8	0.1423	0.0509
1.0	0.1563	0.0537

Table 2: Value of $E(R)$ and $\mathrm{Var}(R)$

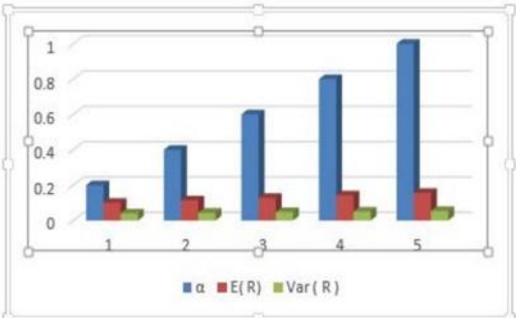

Figure 2: E(R) and Var(R) in Model 3

Section 4: Model 2 with AkashShanker Mixture of $E(\vartheta)$ and $E(3,\vartheta)$ for Kidney Ailment time for treatment and with Exponential Prophylactic time

Assumptions:

(i) Kidney of a patient gets affected at a random time followingAS distribution whose Cdf and pdf are given by (22) and (23).

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(ii) Heart of the patient gets affected in a random time following Erlang distribution with phase 2 and with scale parameter a whose pdf is $f_h(x) = a^2 x e^{-ax}$ and Cdf $F_h(x) = 1 - e^{-ax} - axe^{-ax}$ where $x, a > 0$.

(iii) The patient is also administered prophylactic treatment at a random time according to an exponential distribution with parameter b. This implies that the Cumulative Distribution Function (CDF) and Probability Density Function (PDF) are, respectively, $F_p(x) = 1 - e^{-bx}$; $f_p(x) = be^{-bx}$.

(iv) At time 0 there is no damage to any organ and the random times described above are independent of each other.

(v) The patient is referred for treatment upon the occurrence of either organ failure or for prophylactic purposes, whichever comes first.

(iv) The treatment time for Kidney, Heart or for prophylactic purpose are independent with pdf and Cdf are respectively $(r_k(y), R_k(y))$, $(r_h(y), R_h(y))$ and $(r_p(y), R_p(y))$.

Analysis

Taking into account the occurrence of three failures and three treatments, we observe that the joint probability density function (pdf) of T, representing the time taken to send the patient for hospital treatment, and R, representing the duration of hospital treatment for the patient, is

$$f_{T,R}(x,y) = f_k(x)\overline{F_h}(x)\bar{F}_p(x)r_k(y) + \overline{F_k}(x)f_h(x)\bar{F}_p(x)r_h(y) + \overline{F_k}(x)\overline{F_h}(x)f_p(x)r_p(y).$$

Using Model 2 results we get from (32), the Joint Laplace transform of the pdf is

$$\begin{aligned} f_{T,R}^*(\xi,\eta) =& \left(\frac{\vartheta^3}{\vartheta^2+2}\right)\{\bar{F}_p^*(\xi+\vartheta+a) - a\bar{F}_p^{*\prime}(\xi+\vartheta+a) + \bar{F}_p^{*\prime\prime}(\xi+\vartheta+a) \\ & -a\bar{F}_p^{*\prime\prime\prime}(\xi+\vartheta+a)\}r_k^*(\eta) \\ & +\left\{-a^2\bar{F}_p^{*\prime}(\xi+\vartheta+a) + \left(\frac{a^2}{\vartheta^2+2}\right)\left[2\vartheta\bar{F}_p^{*\prime\prime}(\xi+\vartheta+a) - \vartheta^2\bar{F}_p^{*\prime\prime\prime}(\xi+\vartheta+a)\right]\right\}r_h^*(\eta) \\ & +\left\{1 - \left(\xi + \frac{\vartheta^3}{\vartheta^2+2}\right)\right)\bar{F}_p^*(\xi+\vartheta+a) \\ & +\left[\left(a + \frac{2\vartheta}{\vartheta^2+2}\right)\xi + \frac{a}{\vartheta^2+2}(\vartheta^3+\vartheta^2 a + 2a)\right]\bar{F}_p^{*\prime}(\xi+\vartheta+a) \\ & +\left(\frac{1}{\vartheta^2+2}\right)[-\vartheta(\vartheta+2a)\xi - \vartheta^3 - 2a^2\vartheta]\bar{F}_p^{*\prime\prime}(\xi+\vartheta \\ & +a) + \left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2(\xi+\vartheta+a)\bar{F}_p^{*\prime\prime\prime}(\xi+\vartheta+a)\Big\}r_p^*(\eta) \end{aligned}$$

Here using (39) for exponential prophylactic case this reduces to

$$\begin{aligned} f_{T,R}^*(\xi,\eta) =& \left(\frac{\vartheta^3}{\vartheta^2+2}\right)\left\{\frac{1}{b+\xi+\vartheta+a} + a\frac{1}{(b+\xi+\vartheta+a)^2} + \frac{2}{(b+\xi+\vartheta+a)^3} + a\frac{6}{(b+\xi+\vartheta+a)^4}\right\}r_k^*(\eta) \\ & +\left\{a^2\left(\frac{1}{(b+\xi+\vartheta+a)^2}\right) + \left(\frac{a^2}{\vartheta^2+2}\right)\left[\vartheta\frac{4}{(b+\xi+\vartheta+a)^3} + \vartheta^2\frac{6}{(b+\xi+\vartheta+a)^4}\right]\right\}r_h^*(\eta) \\ & +\left\{1 - \left(\xi + \frac{\vartheta^3}{\vartheta^2+2}\right)\frac{1}{b+\xi+\vartheta+a} + \left[\left(a + \frac{2\vartheta}{\vartheta^2+2}\right)\xi\frac{a}{\vartheta^2+2}(\vartheta^3+\vartheta^2 a+2a)\right]\frac{1}{(b+\xi+\vartheta+a)^2}\right. \\ & +\left(\frac{1}{\vartheta^2+2}\right)\left[-\vartheta(\vartheta+2a)\xi - \vartheta^3 - 2a^2\vartheta\right]\frac{2}{(b+\xi+\vartheta+a)^3} - \left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2(\xi+\vartheta+a)\frac{6}{(b+\xi+\vartheta+a)^4}\Big\}r_p^*(\eta) \end{aligned} \quad (48)$$

The Laplace transform of the marginal distributions of T and R can be written.

$$\begin{aligned} f_T^*(\xi) =& f_{T,R}^*(\xi,0) \\ =& 1 - \xi\left[\frac{1}{b+\xi+\vartheta+a} + \left(a^2 + \frac{2}{\vartheta^2+2}\right)\frac{1}{(b+\xi+\vartheta+a)^2}\right. \end{aligned}$$

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$$+\left(\frac{\vartheta(\vartheta+2a)}{\vartheta^2+2}\right)\frac{2}{(b+\xi+\vartheta+a)^3}+\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2\frac{6}{(b+\xi+\vartheta+a)^4}\Bigg]. \tag{49}$$

$E(T)$ may be written using (49).

$$E(T)=\frac{1}{b+\vartheta+a}+\frac{1}{(b+\vartheta+a)^2}\left(a+\frac{2\vartheta}{\vartheta^2+2}\right)+\frac{2}{(b+\vartheta+a)^3}\left(\frac{\vartheta(\vartheta+2a)}{\vartheta^2+2}\right)+\frac{6}{(b+\vartheta+a)^4}\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2. \tag{50}$$

We find from (49), $E(T^2)$ may be written.

$$E(T^2)=2\frac{1}{(b+\vartheta+a)^2}+\frac{4}{(b+\vartheta+a)^3}\left\{a+\frac{2\vartheta}{\vartheta^2+2}\right\}+\frac{12}{(b+\vartheta+a)^4}\left(\frac{\vartheta(\vartheta+2a)}{\vartheta^2+2}\right)+\frac{48}{(b+\vartheta+a)^5}\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2. \tag{51}$$

This gives

$$\mathrm{Var}(T)=E(T^2)-E(T)^2 \tag{52}$$

gives the variance.

a	E(R)	Var(R)
0.1	0.4778	0.417
0.2	0.4988	0.4245
0.3	0.5198	0.4311
0.4	0.5407	0.4368
0.5	0.5617	0.4417

Table 3: Value of E(T) and Var(T)

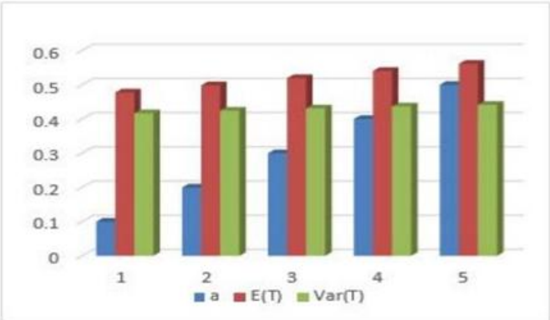

Figure 3: E(T) and Var(T) in Model 4

The Laplace transform of the pdf R can be seen as follows from (48).

$$\begin{aligned} f_R^*(n) &= f_{T,R}^*(0,\eta) \\ &= \left(\frac{\vartheta^3}{\vartheta^2+2}\right)\left\{\frac{1}{b+\vartheta+a} + a\frac{1}{(b+\vartheta+a)^2} + \frac{2}{(b+\vartheta+a)^3} + a\frac{6}{(b+\vartheta+a)^4}\right\} r_k^*(n) \\ &\quad + \left\{a^2\left(\frac{1}{(b+\vartheta+a)^2}\right) + \left(\frac{a^2}{\vartheta^2+2}\right)\left[\vartheta\frac{4}{(b+\vartheta+a)^3} + \vartheta^2\frac{6}{(b+\vartheta+a)^4}\right]\right\} r_h^*(\eta) \\ &\quad + \left\{1 - \frac{1}{b+\vartheta+a}\frac{\vartheta^3}{\vartheta^2+2} - a\frac{1}{(b+\vartheta+a)^2}\left(\frac{\vartheta^3}{\vartheta^2+2}+a\right)\right. \\ &\quad \left. - \frac{2}{(b+\vartheta+a)^3}\frac{1}{\vartheta^2+2}\vartheta(\vartheta^2+2a^2) - \frac{6}{(b+\vartheta+a)^4}\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2(\vartheta+a)\right\} r_p^*(\eta). \end{aligned} \tag{53}$$

Now

$$E(R) = -\frac{\partial}{\partial\eta} f_{T,R}^*(\xi,\eta)_{(0,0)}$$

$E(R)$ may be written and simplified as follows.

$$\begin{aligned} E(R) &= \left(\frac{\vartheta^3}{\vartheta^2+2}\right)\left\{\frac{1}{b+\vartheta+a} + a\frac{1}{(b+\vartheta+a)^2} + \frac{2}{(b+\vartheta+a)^3} + a\frac{6}{(b+\vartheta+a)^4}\right] E(R_k) \\ &\quad + \left\{a^2\left(\frac{1}{(b+\vartheta+a)^2}\right) + \left(\frac{a^2}{\vartheta^2+2}\right)\left[\vartheta\frac{4}{(b+\vartheta+a)^3} + \vartheta^2\frac{6}{(b+\vartheta+a)^4}\right]\right\} E(R_h) \\ &\quad + \left\{1 - \frac{1}{b+\vartheta+a}\frac{\vartheta^3}{\vartheta^2+2} - a\frac{1}{(b+\vartheta+a)^2}\left(\frac{\vartheta^3}{\vartheta^2+2}+a\right)\frac{2}{(b+\vartheta+a)^3}\frac{1}{\vartheta^2+2}\vartheta(\vartheta^2+2a^2)\right. \\ &\quad \left. - \frac{6}{(b+\vartheta+a)^4}\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2(\vartheta+a)\right\} E(R_p). \end{aligned} \tag{54}$$

Similarly, we find

$$\begin{aligned} E(R^2) &= \frac{\partial^2}{\partial\eta^2} f_{T,R}^*(\xi,\eta)_{(0,0)} \\ E(R^2) &= \left(\frac{\vartheta^3}{\vartheta^2+2}\right)\left\{\frac{1}{b+\vartheta+a} + a\frac{1}{(b+\vartheta+a)^2} + \frac{2}{(b+\vartheta+a)^3} + a\frac{6}{(b+\vartheta+a)^4}\right\} E(R_k^2) \\ &\quad + \left\{a^2\left(\frac{1}{(b+\vartheta+a)^2}\right) + \left(\frac{a^2}{\vartheta^2+2}\right)\left[\vartheta\frac{4}{(b+\vartheta+a)^3} + \vartheta^2\frac{6}{(b+\vartheta+a)^4}\right]\right\} E(R_h^2) \\ &\quad + \left\{1 - \frac{1}{b+\vartheta+a}\frac{\vartheta^3}{\vartheta^2+2} - a\frac{1}{(b+\vartheta+a)^2}\left(\frac{\vartheta^3}{\vartheta^2+2}+a\right) - \frac{2}{(b+\vartheta+a)^3}\frac{1}{\vartheta^2+2}\vartheta(\vartheta^2+2a^2)\right. \\ &\quad \left. - \frac{6}{(b+\vartheta+a)^4}\left(\frac{1}{\vartheta^2+2}\right)a\vartheta^2(\vartheta+a)\right\} E(R_p^2). \end{aligned} \tag{55}$$

This gives

$$\mathrm{Var}(R) = E(R^2) - E(R)^2 \tag{56}$$

gives the variance.

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α	E(R)	Var(R)
0.2	0.7325	0.9285
0.4	0.7747	0.9452
0.6	0.8169	0.9665
0.8	0.8591	0.9801
1.0	0.9013	0.9903

Table 4: E(R) and Var(R) in model4

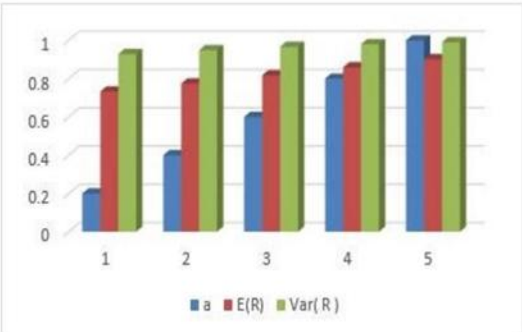

Figure 4: E(R) and Var(R) in model4

CONCLUSION:

In this study, we explored four models involving a diabetic patient with kidney and heart ailments. Each model addressed the timing for hospital treatment, either for kidney or heart issues, or for preventive purposes (prophylactic treatment). The joint probability distribution of the time E(T) to hospital treatment and the treatment duration E(R) was derived, along with their respective probability density functions (pdf) and Laplace transforms.

In Model 1, kidney ailment times follow a mixture of an exponential distribution and an Erlang distribution with phase 3, while heart ailment times follow an Erlang distribution with phase 2. Model 2 builds on Model 1 by incorporating the AkashShanker mixture for kidney ailment times. Model 3 modifies Model 1 by including an exponential distribution for prophylactic treatment times, and Model 4 combines the AkashShanker mixture for kidney ailment times with the exponential prophylactic treatment distribution.

Key observations from our analysis include:

In Model 3 (as shown in Table 1 and Figure 1), increasing the parameter α leads to a reduction in both the expected time (E(T)) and the variance (V(T)) for sending the patient to the hospital. On the other hand (as shown in Table 2 and Figure 2), both the expected treatment time (E(R)) and its variance (V(R)) increase with higher values of α. In Model 4(as shown in Table 3 and Figure 3), an increase in the parameter α leads to an increase in both the expected time E(T) and variance V(T) for sending the patient to the hospital. Similarly(as shown in Table 4 and Figure 4), the expected treatment time E(R) and its variance V(R) also increase with higher values of α. These findings highlight the complex interplay between different distributions and parameters in modelling the timing and duration of hospital treatments for diabetic patients with kidney and heart ailments. Understanding these relationships is crucial for optimizing medical interventions and resource allocation in healthcare settings.

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