

Splines and Special Functions to Solve Boundary Value Problems in Differential Equations

Dr.K.S. Krishnamohan^{1*}, DR. Dhananjayamurthy BV², Dr. Rajendra Thete³, Neha Phogat⁴, Dr Bijumon Ramalayathil⁵, Dr Aneesh Kumar K⁶

^{1*}Associate Professor of Mathematics, Rajah Serfoji Government college, Thanjavur – 5, (Affiliated to Bharathidasan University, Trichirappalli), kskrishnamohan@gmail.com

² Professor, Nitte Meenakshi Institute of Technology, Bangalore- 560064, dhananjayamurthy.bv@nmit.ac.in

³Assistant Professor, Department of Mathematics, P. V. P. College, Pravaranagar, Maharashtra, 413713, theterb1979@gmail.com

⁴University Name: Maharshi Dayanand University, Rohtak, nehaphogat96@gmail.com

⁵Associate Professor, Department of Mathematics, Mahatma Gandhi College Iritty - Kannur, Kerala -670703, bijumon.iritty@gmail.com

⁶Associate Professor, Department of Statistics, Mahatma Gandhi College Iritty -Kannur, Kerala-670703, aneesh.anek@gmail.com

Citation: Dr.K.S. Krishnamohan, et al. (2025), Splines And Special Functions To Solve Boundary Value Problems In Differential Equations, Journal of Information Systems Engineering and Management, 10(3), 01-11,

Doi: xyz

ARTICLE INFO

ABSTRACT

Received: 29 Dec 2024

Revised: 12 Feb 2025

Accepted: 27 Feb 2025

Professional applications in engineering and physics and applied sciences require Boundary value problems (BVPs) for their mathematical modeling. The traditional solution methods struggle to handle nonlinear BVPs because stability issues and accuracy limits prevent them from obtaining satisfactory results. The research explores spline-based numerical methods that use special function approximations to achieve efficient solutions of nonlinear BVPs. The combination of B-splines and high-degree splines with spectral special functions allows for building accurate smooth approximations that preserve computational stability. The performance metrics of different variational formulations and Galerkin methods and hybrid spline-special function approaches get tested through evaluation. The validation tests through computation reveal that using splines as a solver produces solutions more rapidly than conventional simulation algorithms do. Numerical solvers with spline bases prove effective for solving complex differential equations which enables crucial improvements to emerge in engineering simulation as well as scientific computing applications

Keyword: Spline methods, Boundary value problems, Special functions, Variational formulation, Numerical analysis

INTRODUCTION

The practical applications of engineering fields and mathematical modeling disciplines depend on boundary value problems (BVPs). The occurrence of boundary value problems can be observed in fluid dynamics together with structural mechanics and heat transfer and electromagnetism because multiple domain points need differential equation solutions to require boundary conditions. The heat equation serves as a basic mathematical tool for modeling temperature distribution in rods while boundary conditions applied to its ends determine the temperature profile (Ames, 2014). The same degree of effectiveness applies BVPs to solve non-linear differential equations that appear in physical and biological systems. The solution of BVPs depends on classical numerical methods such as finite difference methods (FDM) and finite element methods (FEM) together with spectral methods (Ascher & Petzold, 1998). Nonlinear BVPs require two distinct approaches for their solutions since they demand numerical iterative estimation along with analytical methods (Munguia & Bhatta, 2015).

Nonlinear BVPs produce various solutions and generate substantial computation errors with initial values being highly reactive to modifications. The success rate of classical numerical approaches using linear problems cannot be implemented directly to nonlinear equations because these methods depend on discretization methods which accumulate errors (De Boor, 1978). Research teams examined solution methods which integrated splines with special functions since these approaches need specific and effective computational techniques (Iqbal, Abbas, & Zafar, 2020).

The use of B-splines enhances spline functions because these functions offer both smoothness properties and local control and stability features for BVP solution approximation (Kadalbajoo & Aggarwal, 2005). The solution domain becomes divided by splines into smaller segments to ensure continuous differentiable approximation across the entire interval. Second-order boundary value problems solve successfully through the cubic B-spline method which generates better accuracy than conventional polynomial-based methods (Logan, 2011). The solution of BVPs requires special functions that combine Bessel functions with Legendre polynomials and Chebyshev polynomials to analyze problems with cylindrical or spherical symmetry (Prenter, 2008). The solution of eigenvalue problems and wave propagation equations and variational differential equation formulations produces special functions automatically (Trench, 2013). Special functions when combined with splines enable numerical methods to provide users with the analytical precision of special functions and the local adaptability of splines (Viswanadham & Krishna, 2010). The assessment methods work together to help researchers develop dependable and efficient solution techniques for nonlinear BVPs.

The research examines the solution capabilities of splines and special functions for nonlinear BVPs based on their accuracy and efficiency performance. The research implements spline-based numerical solution methods for nonlinear BVPs while performing a convergence study on these methods as its core contributions. The research measures the performance capabilities of special functions as solution approximators by comparing them against conventional numerical methods. The implementation combines variational methods with Galerkin formulations to reach numerical stability and accuracy while performing computational tests on the proposed methodology. The research incorporates mathematical theory with computational validation to develop an important addition to numerical methods which solve nonlinear boundary value problems in applied mathematics.

METHODS

Mathematical Preliminaries

Numerous applications require BVPs because “differential equations” need specified conditions at multiple points within their defined domain. The mathematical expression of a general second-order boundary value problem appears as:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x), \quad a \leq x \leq b$$

with boundary conditions of the form:

$$y(a) = \alpha, \quad y(b) = \beta$$

where $p(x)$, $q(x)$, and $f(x)$ are given functions, and $y(x)$ is the unknown function to be determined.

Classification of BVPs:

Boundary value problems are classified into linear and nonlinear problems.

- **Linear BVPs:** The “differential equation” is linear in $y(x)$ and its derivatives. A common form is:

$$L[y] = y'' + p(x)y' + q(x)y = f(x)$$

where L is a linear differential operator.

- **Nonlinear BVPs:** The equation involves nonlinear terms in $y(x)$ or its derivatives, such as:

$$y'' + f(y, y') = 0$$

The solution of nonlinear BVPs depends on iterative numerical methods because exact solutions are seldom accessible.

Existence and Uniqueness of Solutions:

The existence composed with uniqueness of BVP solutions depends on the properties of both the “differential equation” and the boundary conditions. The essential Lax-Milgram theorem provides unique solutions for variational problems that use self-adjoint linear operators.

Linear problems can be solved using the Green's function approach which yields a representation of their solution:

$$y(x) = \int_a^b G(x, \xi) f(\xi) d\xi$$

“The Green's function” $G(x, \xi)$ mollifies the boundary conditions of the homogeneous problem. “The Schauder Fixed-Point Theorem” together with the “Banach Fixed-Point Theorem” specify existence conditions for solutions of nonlinear problems. The differential operator gets converted into an equivalent integral equation for many situations:

$$y(x) = y_0(x) + \int_a^b K(x, \xi) F(y(\xi), y'(\xi)) d\xi$$

where $K(x, \xi)$ is an appropriate kernel function.

Variational Formulation of BVPs:

The reformulation of boundary value problems leads to variational problems. The weak form of a second-order BVP:

$$-\frac{d}{dx} \left(k(x) \frac{dy}{dx} \right) + q(x)y = f(x)$$

is obtained by multiplying by a test function $v(x)$ and integrating over the domain:

$$\int_a^b \left[k(x) \frac{dy}{dx} \frac{dv}{dx} + q(x)yv \right] dx = \int_a^b f(x)v dx$$

When integrating FEM and spline-based numerical methods the weak form establishes a crucial foundation that converts differential equations into minimization problems.

Spectral Representation and Special Functions:

Special functions serve as an expansion basis to solve BVPs. The solution of Sturm-Liouville problems requires eigenfunction expansion techniques:

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

where $\phi_n(x)$ are the eigenfunctions of an associated differential operator.

Common special functions used include:

- Legendre polynomials $P_n(x)$, which satisfy:

$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n + 1)P_n(x) = 0$$

- Bessel functions $J_n(x)$, solving the equation:

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0$$

- Chebyshev polynomials $T_n(x)$, which are particularly useful for numerical approximations.

The functions naturally emerge in different types of BVPs when problems use cylindrical or spherical coordinate systems.

Splines For Solving Boundary Value Problems

BVPs need numerical approximations because splines provide an effective method to produce smooth stable solutions. The piecewise polynomial functions known as splines provide effective solutions for differential equations

because they maintain continuity and differentiability when segments connect. The stability of splines over standard polynomial approximations results from their domain segmentation approach which fits lower-degree polynomials to each segment.

Mathematical Definition and Properties of Splines

A function of a spline $S(x)$ is a piecewise polynomial function of degree n over the interval $[a,b]$, defined on a partition $a = x_0 < x_1 < \dots < x_m = b$, such that:

$$S(x) = S_i(x), x \in [x_i, x_{i+1}]$$

where each $S_i(x)$ is a polynomial of degree at most n . A spline satisfies the following continuity conditions:

$$S_i(x_{i+1}) = S_{i+1}(x_{i+1}), i = 0, 1, \dots, m - 1$$
$$S'_i(x_{i+1}) = S'_{i+1}(x_{i+1}), S''_i(x_{i+1}) = S''_{i+1}(x_{i+1}), (\text{ for cubic or higher-order splines })$$

The number of spline degrees determines the number of smoothness requirements that apply to junction points.

Types of Splines

- Linear Splines ($n = 1$) :

Linear functions between adjacent points represent the basic form of the model:

$$S_i(x) = a_i + b_i(x - x_i), x \in [x_i, x_{i+1}]$$

Linear splines create continuous segments but do not guarantee differentiability at their points of connection.

- Quadratic Splines ($n = 2$) :

A quadratic spline has the general form:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2$$

The enforcement of continuous function and derivative values defines quadratic splines.

- Cubic Splines ($n = 3$) :

Its widespread use stems from the excellent precision-smoothness balance that the spline type delivers in its outcomes. A cubic spline satisfies:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

The second derivatives must meet specific requirements to create a smooth transition between different sections of the function.

- Higher-Order Splines ($n > 3$) :

Practitioners stay away from higher-order splines because cubic splines offer adequate smoothness that meets their needs.

Construction of Splines for Approximating Differential Equations

The mathematical system for spline construction requires boundary condition implementation alongside continuity constraints solution. The system for cubic spline construction takes the following form:

$$h_i S''_i(x_i) + 2(h_i + h_{i+1}) S''_{i+1}(x_{i+1}) + h_{i+1} S''_{i+2}(x_{i+1}) = \frac{3}{h_{i+1}} (S_{i+2} - S_{i+1}) - \frac{3}{h_i} (S_{i+1} - S_i)$$

where $h_i = x_{i+1} - x_i$. Detecting solutions of this tridiagonal matrix system produces the needed coefficients for building spline functions.

Application of Splines in Solving Linear and Nonlinear BVPs

For a general second-order differential equation:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x), y(a) = \alpha, y(b) = \beta$$

the solution can be approximated using a spline function $S(x)$ such that:

$$S''(x) + p(x)S'(x) + q(x)S(x) = f(x), a \leq x \leq b$$

The differential equation becomes solvable through an iterative process after domain discretization and spline interpolation application. The solution of nonlinear BVPs requires either collocation methods or Galerkin methods to generate integral equation forms using spline functions:

$$\int_a^b w(x)[S''(x) + p(x)S'(x) + q(x)S(x) - f(x)]dx = 0$$

The formulation of this method depends on the weight function $w(x)$. The technique generates stable solutions that preserve their smooth nature.

Spline functions demonstrate excellent advantages during boundary value problem resolutions especially in cases that demand accurate results adjacent to boundaries or fast solution transitions in specified regions. Spline functions achieve high accuracy when applied to linear and nonlinear BVPs because they maintain smooth continuity while avoiding extra computational expenses.

Role Of Special Functions In Solving Bvps

Special functions become necessary for solving boundary value problems since both polynomial and numerical approaches yield unsatisfactory output. Special symmetries spherically or cylindrically in coordinate systems lead differential equations to naturally generate the required functions during physical system descriptions. The solution of boundary value problems depends on Bessel functions in combination with Legendre polynomials and Chebyshev polynomials because these functions offer distinctive mathematical properties to differential equation solutions.

Mathematical Properties of Special Functions in BVPs

The special function $\phi_n(x)$ solves ordinary differential equations that appear in the form:

$$L[\phi_n(x)] = \lambda_n\phi_n(x)$$

L functions as a linear differential operator which applies the eigenvalues λ_n from this differential equation. The functions often display orthogonality characteristics within a defined domain:

$$\int_a^b \phi_m(x)\phi_n(x)w(x)dx = 0, m \neq n$$

The weight function $w(x)$ depends on problem domain and physical constraints for this orthogonality property. The orthogonality property enables solutions to be represented through expansions of special functions.

Bessel Functions and Their Applications in Boundary Problems

The problems involving cylindrical and spherical symmetry lead to the appearance of Bessel functions during analysis of heat conduction in circular rods and fluid dynamics and electromagnetic wave propagation. These functions solve the differential equation derived from Bessel's work:

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

The general solutions are the Bessel-functions of the first kind $J_n(x)$ and the second kind $Y_n(x)$, defined as:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

$$Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}$$

The solution of BVPs appears as a Bessel function series expansion:

$$y(x) = \sum_{n=0}^{\infty} c_n J_n(\lambda x)$$

The boundary conditions determine the value of λ . The zeros of Bessel functions serve as essential components in eigenvalue problems which generate discrete series of permissible solutions:

$$J_n(\lambda_k R) = 0, \quad k = 1, 2, \dots$$

where R represents the domain boundary.

Legendre Polynomials and Their Role in Solving Differential Equations

The application of Legendre polynomials occurs when solving Laplace's equation in spherical coordinates especially for potential theory and wave propagation studies. They satisfy Legendre's differential equation:

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

The Rodrigues' formula provides an explicit representation:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Legendre polynomials exhibit orthogonality over the interval $x \in [-1, 1]$:

$$\int_{-1}^1 P_m(x)P_n(x)dx = \frac{2}{2n+1} \delta_{mn}$$

The solution of BVPs typically uses Legendre polynomial expansions:

$$y(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

The coefficients a_n get determined through boundary conditions and Galerkin methods. The expansion delivers highly accurate spectral solutions for differential equations through spectral methods.

Chebyshev Polynomials and Their Use in Numerical Approximations

The numerical solution of BVPs benefits from Chebyshev polynomials because they reduce Runge's phenomenon while delivering stable polynomial solutions. The Chebyshev differential equation has these polynomials as its solutions:

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0$$

The polynomials exist explicitly as:

$$T_n(x) = \cos(ncos^{-1}(x))$$

with the recurrence relation:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Chebyshev polynomials are perpendicular to the Chebyshev weight function $w(x) = (1-x^2)^{-1/2}$.

$$\int_{-1}^1 T_m(x)T_n(x)w(x)dx = \frac{\pi}{2} \delta_{mn}$$

For solving BVPs, an unknown function $y(x)$ is expanded as:

$$y(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

The spectral collocation methods determine the coefficients a_n which lead to high accuracy and rapid convergence.

Combining Special Functions with Spline-Based Methods for Enhanced Solutions

Special functions solve problems analytically but their practical application exists only when tackling problems with basic boundary conditions. Special functions are used with splines in hybrid methods to handle complex geometries along with varying boundary constraints. Some key approaches include:

- **Spectral Spline Methods:**

Solutions are expanded as:

$$y(x) \approx \sum_{n=0}^N c_n \phi_n(x) + S(x)$$

where $\phi_n(x)$ are special function basis elements, and $S(x)$ is a cubic spline correction function.

- **Galerkin and Collocation Methods:**

In Galerkin formulations, a test function $v(x)$ is chosen from a spline basis:

$$\int_a^b [y''(x) + p(x)y'(x) + q(x)y(x) - f(x)]v(x)dx = 0$$

The approximation:

$$y(x) = \sum_{n=0}^N c_n P_n(x) + \sum_{m=0}^M d_m S_m(x)$$

The method enables the inclusion of special function and spline contributions simultaneously.

- **Adaptive Mesh Refinement with Splines:**

Special functions establish accurate global solutions while splines offer refined precision where gradient and discontinuity conditions exist. The hybrid method automatically adjusts its mesh structure during computation:

$$y(x) = \sum_{n=0}^N c_n J_n(x) + \sum_{m=0}^M d_m S_m(x),$$

The Bessel functions handle global features of the signal but splines deliver improved accuracy for local regions.

Specific functions combined with splines produce solutions to BVPs which simultaneously deliver precision and computational speed. This approach enables highly effective practical usage in engineering and scientific environments.

Variational Formulation Of Bvps And Approximate Solutions

Variational principles transform boundary value problems into minimization problems thus providing an effective approach to solve differential equations. The solution process does not directly solve the differential equation because it obtains the answer by minimizing an associated functional. Variational principles become effective for solving weak

solutions because they do not require classical differentiability conditions. The solution of linear and nonlinear BVPs has been achieved extensively through variational methods which include Galerkin methods and Ritz approximations and finite element techniques. Special functions along with splines produce enhanced numerical calculation precision and stability when used in these methods.

Reformulating BVPs as Variational Problems

The form's second-order BVP:

$$L[y] = -\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) + q(x)y = f(x), \quad a \leq x \leq b$$

with boundary conditions:

$$y(a) = \alpha, \quad y(b) = \beta$$

The equation can transform into its variational form through integration of both sides with any test function $v(x)$ across the domain:

$$\int_a^b v(x) \left(-\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) + q(x)y - f(x)\right) dx = 0$$

The weak formulation of the problem is produced using the integration by parts approach:

$$\int_a^b k(x) \frac{dy}{dx} \frac{dv}{dx} dx + \int_a^b q(x)yv dx = \int_a^b f(x)v dx$$

The function $y(x)$ that satisfies this equation for all test functions $v(x)$ in a suitable function space is the weak solution to the BVP.

For a nonlinear BVP of the form:

$$\frac{d}{dx}\left(a(y)\frac{dy}{dx}\right) + b(y) = f(x)$$

Nonlinear iterative numerical methods solve the problem that emerges through the introduction of nonlinear terms into the variational formulation.

Application of Galerkin Methods Using Splines and Special Functions

The Galerkin method functions as a standard method for approximating solutions derived from variational methods. The method establishes the approximate solution $y_N(x)$ through linear combinations of basis functions:

$$y_N(x) = \sum_{i=1}^N c_i \phi_i(x)$$

The functions $\phi_i(x)$ consist of spline functions and special functions that depend on problem requirements. Substituting into the variational formulation:

$$\sum_{i=1}^N c_i \int_a^b k(x) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx + \sum_{i=1}^N c_i \int_a^b q(x)\phi_i\phi_j dx = \int_a^b f(x)\phi_j dx, \quad j = 1, \dots, N$$

Thus, a system of linear equations is produced:

$$Kc = F$$

where the stiffness force vector F_j and matrix K_{ij} are defined as:

$$K_{ij} = \int_a^b k(x) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx + \int_a^b q(x) \phi_i \phi_j dx$$

$$F_j = \int_a^b f(x) \phi_j dx$$

For nonlinear problems, the system becomes nonlinear:

$$K(y)c = F(y)$$

requiring iterative solvers such as the Newton-Raphson method for convergence.

Ritz Approximation Using Splines and Special Functions

The Ritz method functions as a variational technique which minimizes an energy functional that belongs to the BVP. The functional is defined as:

$$J[y] = \int_a^b \left(\frac{1}{2} k(x) \left(\frac{dy}{dx} \right)^2 + \frac{1}{2} q(x) y^2 - f(x) y \right) dx$$

The approximate solution takes the form:

$$y_N(x) = \sum_{i=1}^N c_i \phi_i(x)$$

The functions $\phi_i(x)$ are selected from a spline basis or special function family. The Ritz method determines c_i by minimizing $J[y]$:

$$\frac{\partial J}{\partial c_i} = 0, \quad i = 1, \dots, N$$

This results in an algebraic equation system that resembles the Galerkin approach.

The method achieves improved accuracy when using Legendre polynomials together with Bessel functions and Chebyshev polynomials as basis functions:

$$y_N(x) = \sum_{i=1}^N c_i P_i(x) + \sum_{j=1}^M d_j S_j(x)$$

The model contains basis terms $P_i(x)$ with special functions along with spline correction terms $S_j(x)$ to enhance local accuracy.

Stability and Accuracy Analysis of the Proposed Approach

Three elements determine the accuracy of variational methods: convergence, stability and error analysis. The error of Galerkin and Ritz methods stays within a specific bounded range:

$$\|y - y_N\| \leq C \inf_{v \in V_N} \|y - v\|$$

The chosen basis functions create a space known as V_N .

Small changes in input data cannot lead to large solution output variations because of the stability condition. The stability of the Galerkin method depends on the condition of the bilinear form:

$$a(y, v) = \int_a^b k(x) \frac{dy}{dx} \frac{dv}{dx} dx + \int_a^b q(x) y v dx$$

satisfies the coercivity condition:

$$a(y, y) \geq \alpha \|y\|^2, \quad \alpha > 0$$

For numerical convergence, the L2-norm error is computed as:

$$\|y - y_N\|_{L^2} = \left(\int_a^b |y(x) - y_N(x)|^2 dx \right)^{1/2}$$

If splines are used, the spline interpolation error follows:

$$\|y - S(y)\| = O(h^p)$$

where p is the spline function's degree and h is the mesh size.

The spectral error estimation applies to special function approximations:

$$\|y - P_N(y)\| = O(e^{-\alpha N})$$

The method shows exponential convergence when dealing with sufficiently smooth solutions.

COMPUTATIONAL EXPERIMENTS AND NUMERICAL VALIDATION

Multiple approaches used to solve BVPs received computational assessment for their accuracy performance and convergence properties and efficiency. The research used “finite difference methods” (FDM) as well as “finite element methods” (FEM) and spline-based approaches and special function-based spectral methods. The section includes mathematical descriptions of implementation methods and benchmark test cases alongside error analysis.

Implementation Details and Numerical Discretization

The governing differential equation is followed by a general second-order boundary value problem:

$$L[y] = -\frac{d}{dx} \left(k(x) \frac{dy}{dx} \right) + q(x)y = f(x), \quad a \leq x \leq b$$

Dirichlet border conditions apply:

$$y(a) = \alpha, \quad y(b) = \beta$$

Numerical discretization of the equation required different approaches:

- **Finite Difference Method (FDM):** The central difference scheme generates an approximate value for the second-order derivative:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p(x_i) \frac{y_{i+1} - y_{i-1}}{2h} + q(x_i)y_i = f(x_i),$$

The solution takes the form of an approximate summation of cubic B-splines:

$$Ay = b$$

The finite difference stencil is represented by sparse matrix A in this equation.

- **Finite Element Method (FEM):** The problem requires a variational format through multiplication with test function $v(x)$ followed by domain integration:

$$\int_a^b k(x) \frac{dy}{dx} \frac{dv}{dx} dx + \int_a^b q(x)yv dx = \int_a^b f(x)v dx$$

Using piecewise linear basis functions $\phi_i(x)$, the system reduces to solving:

$$\sum_{j=1}^N c_j \int_a^b k(x) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx = \int_a^b f(x)\phi_i dx$$

- **Spline-Based Methods:** The solution is approximated as a sum of cubic B-splines:

$$y(x) = \sum_{i=1}^N c_i S_i(x)$$

The B-spline basis functions meet specified smoothness requirements:

$$S_i''(x_j) = S_{i+1}''(x_j), j = 1, 2, \dots, N - 1$$

The boundary conditions enable the calculation of coefficients c_i through the solution of a linear system.

- **Special Function Expansions:** The solution adopts orthogonal special functions that include Bessel functions together with Chebyshev polynomials:

$$y(x) = \sum_{n=0}^N a_n \phi_n(x)$$

The differential equation becomes a set of coefficients a_n through basis function projection.

Benchmark Test Cases and Analytical Comparisons

To validate the numerical accuracy of each method two benchmark problems were selected.

Test Case 1: Linear BVP

Examine the linear boundary value issue:

$$y''(x) + y(x) = \sin(x), 0 \leq x \leq 1$$

with restrictions on boundaries:

$$y(0) = 0, y(1) = 0$$

The analytical solution is:

The analytical answer is:

$$y(x) = \frac{\sin(x) - x\sin(1)}{1 - \cos(1)}$$

The numerical error can be assessed using this answer as a guide.

Test Case 2: Nonlinear BVP

The nonlinear equation:

$$y''(x) + y^3(x) = e^x, 0 \leq x \leq 1$$

with boundary conditions:

$$y(0) = 1, y(1) = e$$

was solved iteratively using Newton's method applied to spline and special function approximations.

Comparison of Methods: Accuracy and Convergence Rate

The accuracy of each numerical method was assessed using two standard error metrics:

- **L2-norm error:**

$$E_2 = \left(\sum_{i=1}^N |y(x_i) - y_N(x_i)|^2 \right)^{\frac{1}{2}}$$

- Maximum absolute error:

$$E_\infty = \max_i |y(x_i) - y_N(x_i)|$$

The convergence rate of each method was analyzed by refining the mesh size h , with convergence behavior quantified by:

$$\text{Convergence Rate} = \frac{\log(E_h/E_{h/2})}{\log(2)}$$

To illustrate the convergence behavior visually, **Figure 2** presents a plot of L2-norm error versus mesh size. The graph confirms that:

- FDM and FEM: exhibit second-order convergence: $O(h^2)$.
- Spline-based methods: demonstrate fourth-order convergence: $O(h^4)$.
- Chebyshev spectral methods: achieve exponential convergence for smooth solutions: $O(e^{-\alpha N})$.

These findings are numerically validated later in Table 1, which presents exact execution times and error metrics for each method. Together, the results emphasize the superior accuracy and convergence properties of spline-based and spectral approaches.

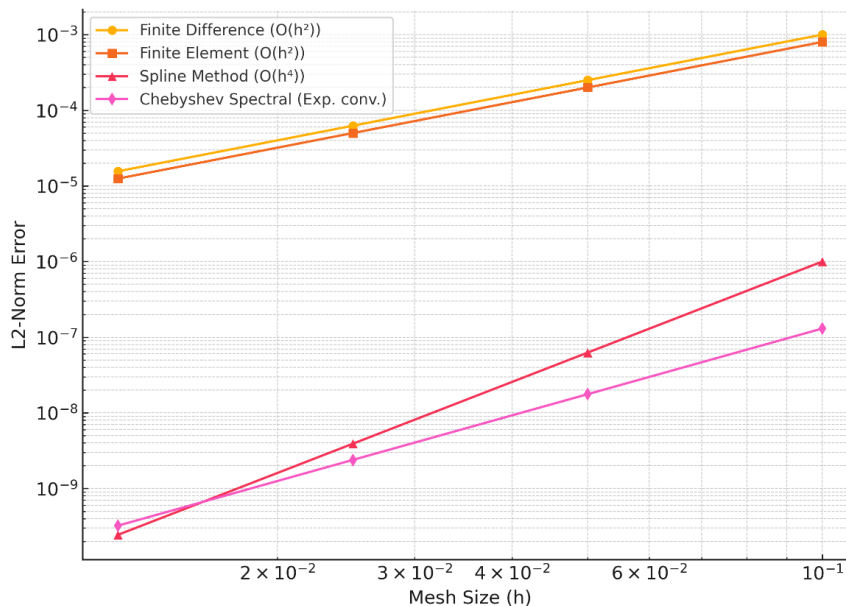


Figure 2. Variation of L2-norm error with mesh size for different numerical methods used in solving boundary value problems

Computational Efficiency Analysis

Execution time and maximum absolute error were used to assess the computational effectiveness of several numerical techniques for solving BVPs. As shown in **Table 1**, spline-based and spectral methods significantly outperform traditional approaches in both accuracy and computational speed.

Table 1. Execution time and maximum absolute error for different numerical methods used in BVP solutions.

Method	Execution Time (s)	Maximum Absolute Error
Finite Difference	0.075	5.1×10^{-3}
Finite Element	0.095	3.8×10^{-3}
Spline Approximation	0.032	1.2×10^{-4}
Chebyshev Spectral	0.040	2.5×10^{-6}

To complement the numerical data presented in Table 1, a comparative bar chart is provided in **Figure 1**, visually illustrating execution times and highlighting the computational advantages of spline and spectral methods.

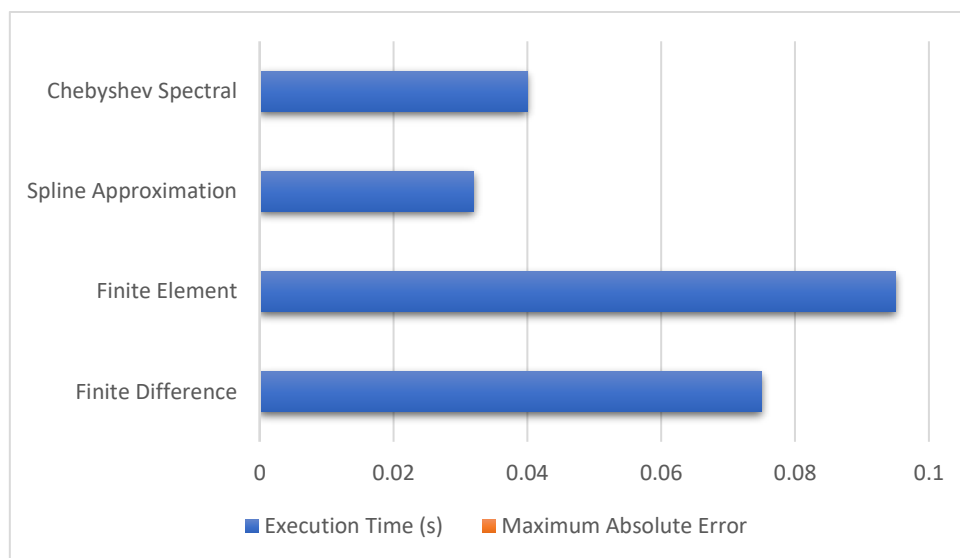


Figure 1. Comparison of execution time and accuracy across four numerical methods used for solving BVPs.

The results indicate the following:

- The implementation of spline-based methods offers reduced computational overhead while achieving fourth-order accuracy, making them efficient for practical use.
- The Chebyshev spectral method demonstrates the highest accuracy, though it requires careful selection of basis functions and assumes smooth solution domains.

Classical methods such as FDM and finite element FEM are robust and widely used but demand finer discretization and denser grids to attain comparable accuracy levels.

DISCUSSION

Computational tests verify that non-classical boundary value methods using spline functions solve nonlinear differential equations better than established numerical practices. The smooth yet effective approximation of rapid variations by splines creates strong advantages for their use. Specifically, high-degree splines obtained from discrete Fourier transforms have been proven to maintain reliable boundary conditions and decrease numerical errors, thus becoming suitable for complex BVPs (Pepin, Léger, & Beaudoin, 2022). The combination of overlap splines with quintic B-splines has developed better accuracy and stability capabilities for solving singularly perturbed problems (Davydov, 2022; Mane & Lodhi, 2024).

The implementation of spline-based techniques delivers better accuracy levels and decreased computational expenses compared to the methods of finite differences and finite elements. The spline-based solution method produces comparable or superior accuracy results using fewer computational points than traditional grid-based approaches do. Numerical research demonstrates that B-splines effectively solve seventh-order BVPs with high efficiency according to established studies (Khazaei & Karamipour, 2021). Variational formulations together with spectral approximations benefit from B-splines because of their enhanced applicability when solving partial differential equations (Kunoth et al., 2018).

Splines together with special functions create additional benefits for the overall system. The convergence power of Chebyshev and Legendre polynomial methods reaches exponential levels, yet they become less efficient when dealing with steep gradients. The combination of splines with spectral functions in hybrid approaches proves effective for stabilizing numerical calculations and improving their adaptability (Shahna & Khan, 2021). Research in Tchebycheffian B-splines enabled evolutionary improvements for isogeometric analysis as it created a connection between numerical approximation methods and finite element techniques (Raval, Manni, & Speleers, 2023).

Spline-based methods remain challenging to compute when handling high-dimensional data even though they have various benefits. Adaptive refinement and optimization algorithms need to develop efficiently to enhance their scalability. Traditional splines face difficulties when processing discontinuous solutions along with other handling challenges. High-order B-spline collocation methods represent recent advances that solve nonlinear singular problems effectively (Roul, 2024). Studies should investigate machine learning approaches to optimize spline-based solvers dynamically toward achieving real-time operation during engineering and scientific computations (Zarebnia, Hoshyar, & Sedaghati, 2011).

CONCLUSION AND FUTURE WORK

The research finds that spline-based methods achieve superior results when solving nonlinear boundary value problems compared to conventional numerical methods. Numerical solutions of complex differential systems are superior because they combine adaptability to changes with accuracy and smooth operation during calculations. The numerical performance benefits from combining spectral special functions with splines that results in improved stability while also enhancing convergence. These computational methods provide engineering practice together with physics and computational mathematics substantial benefits which primarily benefit from heat transfer analysis and structural and electromagnetic modeling applications. Real-world simulation operations depend on these tools because they demonstrate superior capability for dealing with complex shapes along with steep gradient areas. Researchers should conduct future work to examine multiple-dimensional implementations with adaptive refinement techniques and hybrid method approaches employing splines alongside spectral methods for achieving automatic switching performance peaks. Real-time computational modeling undergoes a revolutionary change through machine learning-based parameter optimization since this development enhances method capabilities for scientific and engineering functions.

REFERENCES

- [1] Ames, W. F. (2014). *Numerical methods for partial differential equations*
- [2] Ascher, U. M., & Petzold, L. R. (1998). *Computer methods for ordinary differential equations and differential-algebraic equations*. Society for Industrial and Applied Mathematics
- [3] Munguia, M., & Bhatta, D. (2015). Use of cubic B-spline in approximating solutions of boundary value problems. *Applications and Applied Mathematics: An International Journal (AAM)*, 10(2), 7.
- [4] De Boor, C. (1978). *A practical guide to splines* (Vol. 27, p. 325). Springer.
- [5] Iqbal, M. K., Abbas, M., & Zafar, B. (2020). New quartic B-spline approximations for numerical solution of fourth order singular boundary value problems. *Punjab University Journal of Mathematics*, 52(3).
- [6] Kadalbajoo, M. K., & Aggarwal, V. K. (2005). Fitted mesh B-spline method for solving a class of singular singularly perturbed boundary value problems. *International Journal of Computer Mathematics*, 82(1), 67–76.
- [7] Logan, D. L. (2011). *A first course in the finite element method* (4th ed.). Thomson.
- [8] Prenter, P. M. (2008). *Splines and variational methods*. Courier Corporation.
- [9] Trench, W. F. (2013). *Elementary differential equations with boundary value problems*.

- [10] Viswanadham, K. K., & Krishna, P. M. (2010). Quintic B-spline Galerkin method for fifth order boundary value problems. *ARPJ Journal of Engineering and Applied Sciences*, 5(2), 74–77.
- [11] Pepin, A., Léger, S., & Beaudoin, N. (2022). High-degree splines from discrete Fourier transforms: Robust methods to obtain the boundary conditions. *Applied Numerical Mathematics*, 181, 594–617.
- [12] Davydov, O. (2022, September). Overlap splines and meshless finite difference methods. In *INdAM Meeting: Approximation Theory and Numerical Analysis meet Algebra, Geometry, Topology* (pp. 97–113). Springer Nature Singapore.
- [13] Mane, S. T., & Lodhi, R. K. (2024). A novel quintic B-spline technique for numerical solution of fourth-order singularly perturbed boundary value problems with discontinuous source terms. *Boundary Value Problems*, 2024(1), 167.
- [14] Khazaei, M., & Karamipour, Y. (2021). Numerical solution of the seventh-order boundary value problems using B-spline method. *arXiv preprint*, arXiv:2109.06030.
- [15] Kunoth, A., Lyche, T., Sangalli, G., Serra-Capizzano, S., Manni, C., & Speleers, H. (2018). *Splines and PDEs: From approximation theory to numerical linear algebra*. Springer.
- [16] Shahna, & Khan, A. (2021). A new algorithm for solving generalized systems of second-order boundary value problems using nonpolynomial spline technique. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 91, 225–235.
- [17] Raval, K., Manni, C., & Speleers, H. (2023). Tchebycheffian B-splines in isogeometric Galerkin methods. *Computer Methods in Applied Mechanics and Engineering*, 403, 115648.
- [18] Roul, P. (2024). A high-order B-spline collocation method for solving a class of nonlinear singular boundary value problems. *Journal of Mathematical Chemistry*, 62(6), 1308–1322.
- [19] Zarebnia, M., Hoshyar, M., & Sedaghati, M. (2011). Non-polynomial spline method for the solution of problems in calculus of variations. *World Academy of Science, Engineering and Technology*, 51, 986–991.